Honors Thesis

# Primordial Cosmological Perturbative Modes and the Effect of Strongly Self-Interacting Neutrinos

*Submitted by*

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**Abstract**
As a field, cosmology is defined by the reiterative refinement of cosmological models, with which we make predictions about the history, evolution, and composition of the universe. The current apex of this cycle of refinement is the $\Lambda$ cold dark matter ($\Lambda$CDM) model, primarily due to its incredible ability to model the cosmic microwave background (CMB), one of our most well-understood observables. The $\Lambda$CDM model exhibits some major weaknesses in the form of the Hubble and sigma-8 tensions, and thus we are motivated to seek further refinement. In this thesis, I present a rederivation of the primordial perturbation modes, unifying the scattered and occasionally conflicting literature on the subject. In doing so, I also present a solution for the primordial perturbation modes in the presence of a strongly self-interacting neutrino model. By modifying the dynamics of the early universe, we open the pathway to new research on the effects of this self-interaction on the CMB, the discovery and characterization of physics beyond the Standard Model, and offer a new avenue to attempt to relieve the Hubble and sigma-8 tensions.

# Contents

# 1

# Introduction

Cosmology is the study of the universe as a single unit, and in cosmological research we seek to better understand the history, evolution, and composition of the large-scale structure of the universe. We probe these topics by developing models that can simultaneously agree with the many datasets from current observations of the universe, while also giving explanations to the discrepancies between our predictions and our observations. This recursive self-correction of models is the cycle that cosmologists have used for decades to produce the Lambda cold dark matter ($\Lambda$CDM) model of the universe that is currently favored by most in the field. Barring the current issues of this model (see Perivolaropoulos and Skara, 2022), this $\Lambda$CDM picture does a great job of modeling most of our cosmological data sets, and in particular, does an excellent job of modeling the cosmic microwave background (CMB).

The issues we see in $\Lambda$CDM are deeply concerning because of two major issues (among others; again, see Perivolaropoulos and Skara, 2022), the Hubble and Sigma-8 tensions, point towards discrepancies in two very fundamental parameters. The Hubble tension refers to the fact that using our two main methods of determining the Hubble rate, via the Cosmic Distance Ladder (see Riess et al., 2022) and the predictions of the $\Lambda$CDM model in conjunction with the Planck data (see Aghanim et al., 2020), we get two different values, both with multiple sigma deviation from each other, whose error bars are only shrinking upon refinement of observational methods and the $\Lambda$CDM model. These values seem to diverge upon further measurement, motivating the nomenclature of "tension". The sigma-8 ($\sigma_8$) tension is very similar, this time a measurement of the relative density of the universe on the 8 Mpc scale, which has demonstrated the same diverging behavior between our model-independent probes and the predictions of $\Lambda$CDM. The Hubble rate being the magnitude of expansion for the universe, and $\sigma_8$ being a measure of the anisotropy of the universe, these are two vital measurements for our understanding of the universe, and point towards major issues in our model. See Riess et al., 2022; Di Valentino et al., 2021a; Di Valentino et al., 2021b for in depth conversations about either of these tensions.

These tensions motivate the modification of our current theoretical models as a potential avenue for a resolution. One such modification is the addition of new phenomena to early universe dynamics. This potentially solves two issues; If we include new physics that is beyond the standard model (BSM), and it supplies an improvement of theoretical data's correlation with observational data, we have reason to 1. Investigate more BSM physics as an avenue for improving our theoretical models, and 2. It motivates the belief that our current model, ΛCDM may not be incomplete, but that our understanding of the physics that underlies the ΛCDM model is where our weak point lies.

An example of new physics we could include is a strongly self-interacting neutrino model. This is a particularly enticing approach for two major reasons: the standard model does not predict neutrino self-interaction, making this a BSM phenomenon, and thus a candidate for all of the solutions we discussed previously. The other exciting aspect is that neutrinos are the only cosmological species with two isocurvature modes, which are a type of initial condition, and thus they have two distinct ways that they can affect the predictions of our model. This extra freedom makes neutrinos the best candidate for modification.

The effect of these initial conditions, or adiabatic and isocurvature perturbation modes, on cosmological observables are very well constrained (see Ade et al., 2014; Akrami et al., 2020) by the standard model, and in this model, we expect that the initial conditions of the universe are majority adiabatic in nature, with very tight constraints on the nature and magnitude of isocurvature effects. This means that we have sufficient sensitivity in the tests of our results to identify whether or not the effects of a modified initial condition are realistic candidates for solutions to the issues of the current ΛCDM model. Since these Primordial Cosmological Modes are well characterized and well investigated (see Ma and Bertschinger, 1995; Bucher, Moodley, and Turok, 2000; Piattella, 2018; Baumann, 2023) for current purposes of comparison to observation, the growing Hubble and Sigma-8 tensions can also be resolved by considering the effect of different initial conditions on the evolution of the universe.

In summary, the modification of early universe dynamics offers a unique research avenue by which we may explore the validity of current models without abandoning these models for radical new ideas, offering a deeper understanding of the pitfalls and weaknesses of our current approach.

In this thesis, I prepare a general solution for the perturbative modes in the Conformal Newtonian gauge which is used to derive the evolution equations for all possible initial condition modes. This solution is done taking into account the effects of a strongly self-interacting neutrino model (that of Kreisch, Cyr-Racine, and Doré, 2020) which modifies the evolution of these modes. To obtain this solution, we solve the Einstein and Boltzmann equations in both the standard picture and the system in which we consider neutrino self-interaction. By presenting a solution for the traditional picture, this work also attempts to make obvious and resolve discrepancies in the literature on the subject of the primordial perturbation modes (in the traditional picture without neutrino self-interaction), the primary sources of interest being Ma and Bertschinger, 1995; Bucher, Moodley, and Turok, 2000; Piattella, 2018; Baumann, 2023, a task that informed much of the research required to obtain our result.

# 2

# Background Evolution

In this chapter, we lay the groundwork for the perturbation theory we will use to find our evolution equations. We begin by introducing the concept of the metric and covariance, the most important aspects of General Relativity (GR) for the purposes of this work. We then introduce the cosmological principle and the Friedmann-Lemaître-Robertson-Walker (FLRW) model, which constitutes the underlying assumptions we make to build our background metric. Finally, we briefly talk about the Friedmann equation for our FLRW model and introduce the background metric from which we will build our perturbative model in the following chapter.

## 2.1 A Brief Overview of General Relativity

To do math in cosmology, we have to consider the effects of GR, as on the scales we work with in cosmology, gravity is the dominant force, and our best model for gravity, especially on these scales, is GR. In GR, we model our coordinate system as a continuously connected four-dimensional object called *spacetime*. The shape of spacetime is intrinsically coupled to the mass-energy contained within it, a relation most concisely represented by the Einstein-Field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

The above tensor equation is actually a system of equations described by *the metric tensor*, $g_{\mu\nu}$ which defines the geometry of spacetime, the Ricci tensor $R_{\mu\nu}$ and its associated scalar $R$, which are derived quantities that specifically describe the curvature of spacetime. The right side is described by the *stress-energy tensor*, $T_{\mu\nu}$ which contains all of the mass-energy within the considered region of spacetime. The constants $\Lambda$ and $G$ are the cosmological constant and the Newtonian gravitational constant respectively.

We can give ourselves an intuition of how a metric encodes the geometry of spacetime by writing it out in a matrix representation:

$$g_{\mu\nu} \longrightarrow \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} \longleftrightarrow \begin{pmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ g_{xt} & g_{xx} & g_{xy} & g_{xz} \\ g_{yt} & g_{yx} & g_{yy} & g_{yz} \\ g_{zt} & g_{zx} & g_{zy} & g_{zz} \end{pmatrix},$$

where the second depiction shows which terms describe the couplings between which coordinates. For the sake of familiarity, the coordinates shown are the traditional Cartesian coordinates, time $t$ and space $x, y, z$, but the beauty of this theory comes from the fact that these coordinates can be absolutely anything, as long as they form a proper representation of 4D space.

General relativity as a theory is written in terms of tensors, who possess the extremely favorable mathematical property of *covariance*. Covariance is the fundamental freedom in the choice of coordinate system I just mentioned, and is so favorable because regardless of the fact that two observers in mathematically distinct reference frames will observe mathematically distinct dynamics, when one observer transforms their reference frame to that of the other, their equations and observations will always agree.

## 2.2 Scales and Coordinates in Cosmology

Often in cosmology we assume the *cosmological principle* which is the idea that the universe is homogeneous and isotropic on sufficiently large scales (for a discussion, see section 2.2 of Ryden, 2003). The version of the FLRW model (or metric) that we employ in this work hinges on the cosmological principle as one of two underlying assumptions. The other assumption we make is that the universe is flat, which has been observed to be true, (see Aghanim et al., 2020, specifically their value for $\Omega_K$, shown in the conclusion) at least to the extent of our observation capabilities.

The FLRW model is the closest thing to a global solution for the Einstein equations, as it describes a homogeneous, isotropic, and expanding universe. Obviously, the weak point of this model is that it becomes invalid at scales less than 100 Mpc, where the cosmological principle breaks down, but it provides a way of talking about the universe as a whole, which is of great importance if we ever want to compare our model to observations. For our purposes, the most important feature of the FLRW metric is that it provides a set of mathematical statements that we know to be true on the global scale of the universe.

In the FLRW model we note that the universe is expanding, which is observationally proven and presents an additional challenge for when we attempt to build a coordinate system for our solutions. We can make use of the covariance of GR by redefining our coordinate system in terms of the *scale factor* $a(\tau)$, which denotes the relative expansion of the universe at a given *conformal time* $\tau$, which is the coordinate for time in this new system. By moving to this conformal coordinate system we intrinsically factor in the effect of the expansion into our coordinates. Mathematically we redefine in the following way: Consider the classic four-position, $x^\mu = (t, \boldsymbol{x})$ where the bold represents a vector. We may define the conformal or *comoving* four-position in terms of differentials of the classic four-position, specifically:

$$dx^\mu \rightarrow (dx^\mu)' = \frac{dx^\mu}{a(\tau)},$$

$$\begin{pmatrix} d\tau \\ dx \\ dy \\ dz \end{pmatrix} = \frac{1}{a(\tau)} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}.$$

For the remainder of this text, the comoving coordinates will be denoted as $x^\mu$, and all of our math will be within this conformal coordinate system.

## 2.3 Friedmann Equation

In order to begin talking about perturbations, we need a background from which to perturb our system. As we've stated, the flat FLRW model provides an ideal background from which to perturb since it encapsulates all of our fundamental underlying assumptions about the structure of the universe on a global scale.

The Friedmann equation relates the expansion of the universe to its constituent components. We have matter and radiation, who contribute energy density ($\rho$) to the universe through their mass and energy via the stress-energy tensor. We also have curvature ($\kappa$) and the cosmological constant, both of which contribute to the energy density of the universe through their effect on the geometry. In conformal coordinates, we see the Friedmann equation appear as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G a^2 \rho_{tot}}{3} - \kappa \tag{2.1}$$

where the total energy density can be broken into components based off of the species:

$$\rho_{tot} = \rho_\gamma + \rho_\nu + \rho_c + \rho_b + \rho_\Lambda .$$

The associated flat FLRW background metric, denoted $\bar{g}_{\mu\nu}$, we write as

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right],$$

or in matrix form:

$$\bar{g}_{\mu\nu} \rightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2\sin^2\theta \end{pmatrix},$$

# 3

# Evolution of Perturbations

In this work, we are interested in the ways our metric may be changed from the FLRW to some slightly perturbed form, where these perturbations are being driven by the initial conditions following the big bang and inflation. The condition that the effects we are deriving must come from the inflationary epoch is what motivates us to call the effect primordial, and the nuances and origin of this requirement will be discussed in a later chapter.

In this chapter, we begin by talking about the gauge problem for the Synchronous and Conformal Newtonian gauges, deriving the transformation law to go between them. We follow this by deriving the Einstein equations from the perturbed metric. Finally, we derive the Boltzmann equations for each species as the last step in our preparation for finding the primordial cosmological modes.

Before moving forward, it is important for us to define the two most popular gauges used in finding these modes: First is the **Synchronous** gauge:

$$ds^2 = a^2(\tau)\{-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j\}, \tag{3.1}$$

whose perturbation $h_{ij}$ can be decomposed into scalar, parallel, perpendicular, and transverse components:

$$h_{ij} = \frac{\delta_{ij} h}{3} + h_{ij}^{\parallel} + h_{ij}^{\perp} + h_{ij}^{T}.$$

We will be working in Fourier space, or $k$-space, for the majority of this work, and thus we can define $h_{ij}$ in $k$-space as

$$h_{ij}(\boldsymbol{x}, \tau) = \int d^3k \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left\{ \hat{k}_i \hat{k}_j h(\boldsymbol{k}, \tau) + \left( \hat{k}_i \hat{k}_j - \frac{1}{3}\delta_{ij} \right) 6\eta(\boldsymbol{k}, \tau) \right\}. \tag{3.2}$$

Second, we have the **Newtonian** gauge:

$$ds^2 = a^2(\tau)\{-(1 + 2\Psi)d\tau^2 + (1 + 2\Phi)dx^i dx_i\}, \tag{3.3}$$

where the potentials $\Phi$ and $\Psi$ are the famous Bardeen potentials (derived in Bardeen, 1980, though the notation used here comes from Kodama and Sasaki, 1984).

## 3.1 Gauge Problem

The gauge problem refers to the ambiguity in the choice of coordinate due to the covariance of General Relativity. To resolve this we **fix the gauge**, transforming our observer to an arbitrary, well defined coordinate system, which we do by perturbing the background metric like with some arbitrary perturbing metric $\delta g_{\mu\nu}$:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu},$$
$$g_{\mu\nu} \to a^2(\tau) \begin{pmatrix} -1 - 2\alpha(x^\mu) & w_i(x^\mu) \\ w_i(x^\mu) & \delta_{ij}[1 + 2\beta(x^\mu)] + \chi_{ij}(x^\mu) \end{pmatrix},$$

where $\delta_{ij}$ is the Kronecker delta, and $\chi_{ii} = 0$. Now within this arbitrary metric with modes $\alpha, w_i, \beta, \chi_{ij}$, we want to build an understanding of these arbitrary parameters such that when we choose a particular gauge, we can describe transformations between popular gauges with relative ease. This analysis is done by considering a general transformation:

$$x^\mu \to \hat{x}^\mu = x^\mu + d^\mu(x^\nu)$$

and we start by splitting this into its effect on the timelike and spacelike portions:

$$\hat{x}^0 = x^0 + A(x^\mu),$$
$$\hat{x}^i = x^i + \nabla^i B(x^\mu) + C^i(x^\mu),$$

where we've split the effect on the spacelike portion into a pair of longitudinal ($B$) and transverse ($C^i$) components, given $\nabla_i C^i = 0$. The key result we get by splitting the transformation like this is a generalized transformation law of the metric,

$$\hat{g}_{\mu\nu}(x^\lambda) = g_{\mu\nu}(x^\lambda) - g_{\mu\beta}(x^\lambda)\partial_\nu d^\beta - g_{\alpha\nu}(x^\lambda)\partial_\mu d^\alpha - d^\alpha \partial_\alpha g_{\mu\nu}(x^\lambda) + \mathcal{O}(d^2), \tag{3.4}$$

this is the law for transforming between any pair of gauges, and is where we obtain the law we will use to transform between our two gauges. After a string of analysis that can be found in section 3 of Ma and Bertschinger, 1995, we obtain the following laws for transformation in

$k$-space:

$$\Psi(\boldsymbol{k},\tau) = \frac{1}{2k^2}\left\{\ddot{h}(\boldsymbol{k},\tau) + 6\ddot{\eta}(\boldsymbol{k},\tau) + \frac{\dot{a}}{a}\left[\dot{h}(\boldsymbol{k},\tau) + 6\dot{\eta}(\boldsymbol{k},\tau)\right]\right\}, \tag{3.5}$$

$$\Phi(\boldsymbol{k},\tau) = \frac{1}{2k^2}\frac{\dot{a}}{a}\left[\dot{h}(\boldsymbol{k},\tau) + 6\dot{\eta}(\boldsymbol{k},\tau)\right] - \eta(\boldsymbol{k},\tau). \tag{3.6}$$

From this result, we can reverse engineer a way from the Newtonian gauge to the Synchronous, which will come in handy when we want to analyze our solutions in later work:

$$-\dot{\eta} + \left(\frac{\ddot{a}}{a} - \frac{\dot{a}}{a}\left(1 + \frac{\dot{a}}{a}\right)\right)\eta = \dot{\Phi} - \frac{\dot{a}}{a}\Psi + \left(\frac{\dot{a}}{a}\left(1 + \frac{\dot{a}}{a}\right) - \frac{\ddot{a}}{a}\right)\Phi. \tag{3.7}$$

From here on, we will be working in the Newtonian gauge unless otherwise stated.

## 3.2 Einstein Equations

The next step in our solution is solving the perturbed part of the Einstein equations, or the *linearized* Einstein equations, which can be written in terms of the background and perturbation of the metric:

$$\bar{g}^{\mu\rho}\delta R_{\rho\nu} + \delta g^{\mu\rho}\bar{R}_{\rho\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}\delta R = 8\pi G\delta T^{\mu}{}_{\nu}\,. \tag{3.8}$$

where the Ricci tensor can be broken into it's background and perturbed components the same way we have done with the metric.

Before we move forward, we need to quickly write down our stress-energy tensor for context of the parameters we will be solving for in the upcoming chapters and sections:

$$\begin{aligned} T^{0}{}_{0} &= -(\bar{\rho} + \delta\rho), \\ T^{0}{}_{i} &= (\bar{\rho} + \bar{P})v_i, \\ T^{i}{}_{j} &= (\bar{P} + \delta P)\delta^{i}{}_{j} + \Sigma^{i}{}_{j}, \quad \Sigma^{i}{}_{i} = 0, \end{aligned} \tag{3.9}$$

these quantities will later be recast into our perturbative variables,

$$(\bar{\rho} + \bar{P})\theta = ik^j\delta T^{0}{}_{j}\,, \qquad (\bar{\rho} + \bar{P})\sigma = -\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)\Sigma^{i}{}_{j},$$

when we solve for our evolution equations. In general $\rho_i$ and $P_i$ are the density and pressure of some species $i$, and $\delta_i = \delta\rho_i/\rho_i$ where $\rho_i = \bar{\rho}_i + \delta\rho_i$ is the decomposition into the background and perturbed contributions to a given parameter, in this case density. The perturbative parameters we just defined, $\theta_i$ and $\sigma_i$ are the velocity and shear-stress of some species $i$, which do not have the same decomposition since they are purely perturbative quantities.

We obtain four equations from the linearized Einstein equations:

$$-k^2\Phi + \frac{3\dot{a}}{a}\left(\frac{\dot{a}}{a}\Psi - \dot{\Phi}\right) = 4\pi G a^2 \delta T^0{}_0\,, \tag{3.10}$$

$$k^2\left(\frac{\dot{a}}{a}\Psi - \dot{\Phi}\right) = 4\pi G a^2 (\bar{\rho} + \bar{P})\theta, \tag{3.11}$$

$$-\ddot{\Phi} + \frac{\dot{a}}{a}\left(\dot{\Psi} - 2\dot{\Phi}\right) + \left(2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\Psi - \frac{k^2}{3}(\Phi + \Psi) = \frac{4\pi}{3} G a^2 \delta T^i{}_i\,, \tag{3.12}$$

$$-k^2(\Phi + \Psi) = 12\pi G a^2 (\bar{\rho} + \bar{P})\sigma. \tag{3.13}$$

The last piece of information we need from the Einstein equations actually comes from the conservation law of energy-momentum for some general fluid (where in GR, a fluid refers to some continuous, deformable object with mass-energy, density, and pressure),

$$\nabla_\mu T^{\mu\nu} = 0 = \partial_\mu T^{\mu\nu} + \Gamma^\nu{}_{\alpha\beta} T^{\alpha\beta} + \Gamma^\alpha{}_{\alpha\beta} T^{\nu\beta},$$

from which we can gleam an understanding of the *fractional density perturbation*, $\delta = \delta\rho/\rho$ and velocities $\theta$ in our stress-energy:

$$\dot{\delta} = -(1+w)(\theta + 3\dot{\Phi}) - 3\frac{\dot{a}}{a}\left(\frac{\delta P}{\delta\rho} - w\right)\delta, \tag{3.14}$$

$$\dot{\theta} = -\frac{\dot{a}}{a}(1-3w)\theta - \frac{\dot{w}}{1+w}\theta + \frac{\delta P/\delta\rho}{1+w}k^2\delta - k^2\sigma + k^2\Psi. \tag{3.15}$$

The transformations for the stress-energy between our two gauges is relatively simple: with the Synchronous gauge on the left, and the Newtonian on the right, we find that

$$\hat{\delta} = \delta - \frac{1}{2k^2}\frac{\dot{\bar{\rho}}}{\bar{\rho}}(\dot{h} + 6\dot{\eta}), \tag{3.16}$$

$$\hat{\theta} = \theta - \frac{1}{2}(\dot{h} + 6\dot{\eta}), \tag{3.17}$$

$$\delta\hat{P} = \delta P - \frac{1}{2k^2}\dot{\bar{P}}(\dot{h} + 6\dot{\eta}), \tag{3.18}$$

$$\hat{\sigma} = \sigma. \tag{3.19}$$

These relations will come in handy much later for computation, and are vital for future work regarding implementation in Boltzmann solvers, such as CLASS (Lesgourgues, 2011) or CAMB (Lewis and Challinor, 2011).

## 3.3 Boltzmann Equations

The last step in our preparation is done by evaluating the Boltzmann equations to obtain evolution equations for the species in our system. These equations describe any general body, not just a fluid like eqs. (3.14) and (3.15), making them much more general, and able to describe much more complex systems. We start by taking the spacial portion of the conjugate momentum of our system to be $P_i$, expressed in terms of the proper 4-momentum $p_\mu$:

$$P_i = a(1 + \Phi)p_i,$$

we then specify that the differential number of particles $dN$ belonging to some differential volume of our phase space is

$$f(x^i, P_j, \tau)dx^1dx^2dx^3dP_1dP_2dP_3 = dN,$$

where $f$ is the phase space distribution who is scalar and generally invariant under transformation. We can then write our stess-energy in terms of the conjugate momentum and the distribution,

$$T^\mu{}_\nu = \int dP_1dP_2dP_3\frac{P^\mu P_\nu}{\sqrt{-g}P^0}f(x^i, P_j, \tau), \tag{3.20}$$

where

$$f(x^i, P_j, \tau) = f_0(ap)[1 + \mathcal{F}(x^i, P_j, \tau)],$$

where $f_0$ is the Fermi-Dirac or Bose-Einstein Distribution, and $\mathcal{F}$ is the perturbation to the distribution due to the metric. We introduce a new parameter, the comoving momentum $q_i = ap_i$, with decomposition $q_i = qn_i$ where $n_i$ is a direction, to remove the coordinate dependence on the perturbation from our phase space, and we are left with

$$f(x^i, q_j, \tau) = f(q)[1 + \mathcal{F}(x^i, q_j, \tau)].$$

The total derivative of our phase space is our Boltzmann Equation,

$$\frac{Df}{d\tau} = \frac{\partial f}{\partial \tau} + \frac{dx^i}{d\tau}\frac{\partial f}{\partial x^i} + \frac{dq}{d\tau}\frac{\partial f}{\partial q} + \frac{dn_i}{d\tau}\frac{\partial f}{\partial n_i} = C[f], \tag{3.21}$$

which simplifies to

$$\frac{\partial \mathcal{F}}{\partial \tau} + i\frac{q}{\epsilon}(\boldsymbol{k}\cdot\hat{n})\mathcal{F} - \frac{d\ln f_0}{d\ln q}\left[\dot{\Phi} + i\frac{\epsilon}{q}(\boldsymbol{k}\cdot\hat{n})\Psi\right] = \frac{1}{f_0}C[f], \tag{3.22}$$

where $\epsilon = \sqrt{q^2 + a^2m^2}$ is the comoving energy. The right term $C[f]$ is the collision term, capturing any interactions that we may include in our model. The specifics of this term are very unique to the situation and species being discussed, and will come into play when we start talking about our evolution equations for photons and self-interacting neutrinos.

From our description of the stress-energy in eq. (3.20) and its components in eqs. (3.14) and (3.15), we can obtain evolution equations for each species.

### 3.3.1 Cold Dark Matter

The equation of state for cold dark matter (CDM) is $w = c_s^2 = 0 \implies \dot{w} = 0$, and only interacts via gravity, implying $\sigma = 0$, thus we are left with

$$\dot{\delta}_c = -\theta_c - 3\dot{\Phi}, \tag{3.23}$$

$$\dot{\theta}_c = -\frac{\dot{a}}{a}\theta_c + k^2\Psi. \tag{3.24}$$

### 3.3.2 Massless Neutrinos

Massless neutrinos have the unique property that $\epsilon = q$, which greatly simplifies the Boltzmann equation. In the early universe, neutrinos are massless and thus highly relativistic, and so their energy density is very simply related to their pressure via their equation of state, $w = 1/3$, giving us the relation $\rho_\nu = 3P_\nu$. From eq. (3.20), we can find three major equations for the energy density $\delta\rho_\nu$, pressure $\delta P_\nu$, energy flux $\left(\delta T^0{}_i\right)_\nu$, and shear stress $\left(\Sigma^i{}_j\right)_\nu = \left(T^i{}_j\right)_\nu - P_\nu\delta_{ij}$:

$$\delta\rho_\nu = 3\delta P_\nu = a^{-4}\int q^2 dq d\Omega\, q f_0(q)\mathcal{F}, \tag{3.25}$$

$$\left(\delta T^0{}_i\right)_\nu = a^{-4}\int q^2 dq d\Omega\, q n_i f_0(q)\mathcal{F}, \tag{3.26}$$

$$\left(\Sigma^i{}_j\right)_\nu = a^{-4}\int q^2 dq d\Omega\, q\left(n_i n_j - \frac{1}{3}\delta_{ij}\right) f_0(q)\mathcal{F}, \tag{3.27}$$

we then integrate away our $q$ dependence and expand the angular dependence in terms of the Legendre polynomials $P_\ell(\hat{k}\cdot\hat{n})$,

$$F_\nu(\boldsymbol{k},\hat{n},\tau) = \frac{\int q^2 dq\, q f_0(q)\mathcal{F}}{\int q^2 dq\, q f_0(q)} = \sum_{\ell=0}^{\infty}(-i)^\ell(2\ell+1)F_{\nu,\ell}(\boldsymbol{k},\tau)P_\ell(\hat{k}\cdot\hat{n}).$$

and thus we can recast our the parameters of our stress-energy (eq. (3.9)) in terms of our perturbative variables, and the expansion in $\ell$, called the multipole expansion (this same expansion method is used on photons in the next section):

$$\begin{aligned}\delta_\nu &= F_{\nu,0},\\ \theta_\nu &= \frac{3}{4}kF_{\nu,1},\\ \sigma_\nu &= \frac{1}{2}F_{\nu,2}.\end{aligned}$$

We can then integrate eq. (3.22) through $q^2 dq\, q f_0(q)$ and divide by $\int q^2 dq\, q f_0(q)$ to recast our Boltzmann equation in the massless neutrino context,

$$\frac{\partial F_\nu}{\partial\tau} + ik\mu F_\nu = -4(\dot{\Phi} + ik\mu\Psi)$$

then using our Legendre expansion as well as the orthogonality condition of the polynomials to obtain our evolution equations:

$$\dot{\delta}_\nu = -\frac{4}{3}\theta_\nu - 4\dot{\Phi}, \tag{3.28}$$

$$\dot{\theta}_\nu = k^2\left(\frac{1}{4}\delta_\nu - \sigma_\nu\right) + k^2\Psi, \tag{3.29}$$

$$\dot{F}_{\nu,\ell} = \frac{k}{2\ell+1}[\ell F_{\nu,(\ell-1)} - (\ell+1)F_{\nu,(\ell+1)}], \qquad \ell \geq 2. \tag{3.30}$$

### 3.3.3 Photons

Photons have unique evolution equations before and after recombination. Before recombination, we see photons and baryons couple tightly, interacting via Thompson scattering. This is a non-conservative process from the perspective of our Boltzmann equation since it involves energy transfer between two different species, thus we expect this collisional term to effect all multipoles $\ell$. This means we expect a new collisional term to appear on the right side of our Boltzmann equation. This collisional term is very complicated, having a familiar $F_\gamma$ term that denotes the phase space density perturbation, similar to its role in the neutrino picture, and a new term $G_\gamma$ that handles the polarization perturbation due to the scattering. Both terms are expanded through a multipole method to give the same hierarchical structure that we obtained in the neutrino picture. The results of this analysis reveal the following solutions, (see Ma and Bertschinger, 1995 for description of this process)

$$\dot{\delta}_\gamma = -\frac{4}{3}\theta_\gamma - 4\dot{\Phi} \tag{3.31}$$

$$\dot{\theta}_\gamma = k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + k^2\Psi + an_e\sigma_T\left(\theta_b - \theta_\gamma\right) \tag{3.32}$$

$$\dot{F}_{\gamma 2} = 2\dot{\sigma}_\gamma = \frac{8}{15}\theta_\gamma - \frac{3}{5}kF_{\gamma 3} - \frac{9}{5}an_e\sigma_T\sigma_\gamma + \frac{1}{10}an_e\sigma_T\left(G_{\gamma 0} + G_{\gamma 2}\right) \tag{3.33}$$

$$\dot{F}_{\gamma\ell} = \frac{k}{2\ell+1}\left[\ell F_{\gamma(\ell-1)} - (\ell+1)F_{\gamma(\ell+1)}\right] - an_e\sigma_T F_{\gamma\ell}, \quad \ell \geq 3 \tag{3.34}$$

$$\dot{G}_{\gamma\ell} = \frac{k}{2\ell+1}\left[\ell G_{\gamma(\ell-1)} - (\ell+1)G_{\gamma(\ell+1)}\right] + an_e\sigma_T\left[-G_{\gamma\ell} + \frac{1}{2}\left(F_{\gamma 2} + G_{\gamma 0} + G_{\gamma 2}\right)\left(\delta_{\ell 0} + \frac{\delta_{\ell 2}}{5}\right)\right]. \tag{3.35}$$

After recombination, photons become essentially transparent to baryons, and scattering becomes a less noticeable effect, though it still is non-negligible.

### 3.3.4 Baryons

Baryons (and leptons), in absence of their tight coupling to photons, behave as a non-relativistic fluid and their evolution can be found similarly to the process for cold dark matter: we see that $w \approx c_s^2 = 0$ unless the term is already on large scales $\propto k^2$, in which case the squared sound speed $c_s^2 > 0$ becomes non-negligible. We see this complication appear in our evolution equations:

$$\dot{\delta}_b = -\theta_b - 3\dot{\Phi}, \tag{3.36}$$

$$\dot{\theta}_b = -\frac{\dot{a}}{a}\theta_b + c_s^2 k^2 \delta_b + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e \sigma_T (\theta_\gamma - \theta_b) + k^2\Psi, \tag{3.37}$$

along with the perturbation on the velocity due to the Thomson scattering process between radiation. A more in depth conversation of the scattering term can be found in Ma and Bertschinger, 1995.

# 4

# Primordial Cosmological Modes

In this chapter we finally derive the primordial cosmological modes for which we have been working, not before we discuss initial conditions and the theory that validates these modes as physical solutions, as opposed to just quirks of the gauge used. We also define and discuss the different types of primordial modes.

## 4.1 Super-horizon Initial Conditions

Before we state the various primordial cosmological modes we can solve for from our work, we need one last set of equations stemming from the initial conditions of the universe. At first this is a daunting task, but using the inflationary theory of the universe, we can give a description of the origin of these initial conditions via super-horizon scale perturbations.
During inflation, quantum fluctuations purvey the quark-gluon plasma of the early universe, during which we have an exponential expansion rate, much faster than that of speed of light. As this expansion rate grows, the scale and magnitude of these quantum fluctuations balloon, exiting the causal horizon, becoming *super-horizon* in scale. In $k$ space, we denote this regime as the period in which $\tau \ll 1/k$. Once inflation slows and the causal horizon catches up, we see these fluctuations re-enter the causal universe in the *horizon crossing* regime, $1 \approx k\tau$. It is in this crossing regime that the initial conditions of the universe are set, since the formerly small quantum perturbations re-enter the universe and begin effecting the densities and velocities of the species within. Since we are so deep in the radiation dominated epoch, we can make some sweeping approximations going forward:

$$a(\tau) \approx \sqrt{\Omega_{r,0}} H_0 \tau, \quad \frac{\dot{a}}{a} \approx \frac{1}{\tau}$$

where $H_0$ is the non-conformal Hubble rate today. Then, from eq. (2.1), we have

$$4\pi G a^2 = \frac{3\dot{a}^2}{2a^2 \rho_{\text{tot}}}, \tag{4.1}$$

and we can make the claim that the density of the cosmological constant $\rho_\Lambda \ll \rho_\gamma + \rho_\nu + \rho_c + \rho_b$ such that it is negligible in the total density $\rho_{\text{tot}}$. Moreover, if we define the density fraction

$R_i = \rho_i/\rho_{\text{tot}}$, then

$$R_\gamma + R_\nu + R_c + R_b = 1$$

and furthermore $R_c + R_b \ll R_\gamma + R_\nu$ since we are so deep in radiation domination, thus $R_\gamma + R_\nu \approx 1$. Another observation we should make is the hierarchical order that appears from multipole expansion, where each term $F_{i,\ell} = \mathcal{O}(k\tau)^\ell$, and since we know $k\tau$ is very small in the super-horizon picture, we learn from the linearized Einstein equations that

$$\begin{aligned}
\Phi &\sim \Psi \sim \delta_i, \\
\theta_i &\propto F_{i,1} \sim (k\tau)\Phi \sim (k\tau)\Psi, \\
\sigma_i &\propto F_{i,2} \sim (k\tau)^2\Phi \sim (k\tau)^2\Psi,
\end{aligned}$$

and terms that are order zero in $k\tau$ dominate those that are order one, and those that are order one dominate those that are order two, etc. We can use this approximation to argue that from eq. (3.14), we see the simple relation

$$\dot{\delta}_\gamma = -4\dot{\Phi}, \quad \dot{\delta}_\nu = -4\dot{\Phi}, \quad \dot{\delta}_c = -3\dot{\Phi}, \quad \dot{\delta}_b = -3\dot{\Phi},$$

and after integration,

$$\delta_\gamma = -4\Phi + 4C_\gamma, \quad \delta_\nu = -4\Phi + C_\nu, \quad \delta_c = -3\Phi + C_c, \quad \delta_b = -3\Phi + C_b,$$

which we then recast as

$$\delta_\gamma = -4\Phi + 4C_\gamma, \tag{4.2}$$

$$\delta_\nu = \delta_\gamma + S_\nu, \tag{4.3}$$

$$\delta_c = \frac{3}{4}\delta_\gamma + S_c, \tag{4.4}$$

$$\delta_b = \frac{3}{4}\delta_\gamma + S_b, \tag{4.5}$$

with

$$S_\nu = C_\nu - 4C_\gamma, \quad S_c = C_c - 4C_\gamma, \quad S_b = C_b - 4C_\gamma,$$

where $C_\gamma$ is the **adiabatic mode**, and $S_i$ are the **density isocurvature modes** who we will discuss in depth in the following section. These modes are functions of $\boldsymbol{k}$ but not $\tau$.

The final key approximation we can make as a result of the time period we are in is the **tight-coupling** approximation, where due to the tight coupling of baryons and photons, the Thompson drag term $an_e\sigma_T$ explodes since baryons and photons essentially form a single fluid. In this approximation, we see $\theta_\gamma \approx \theta_b$, and since the scales of these perturbations are so large, the term $c_s^2k^2\delta_b$ in eq. (3.36) will drop since $c_s^2k^2 \approx 0$, which effectively recasts the velocity evolution of baryons to

$$\dot{\theta}_b = -\frac{\dot{a}}{a}\theta_b + k^2\Psi,$$

which we should notice is totally equivalent to that of CDM, indicating that we can argue that the difference $\dot{\theta}_b - \dot{\theta}_c$:

$$\begin{aligned}\frac{d}{d\tau}(\tau\theta_b) &= \frac{d}{d\tau}(\tau\theta_c),\\ \theta_\gamma = \theta_b &= \theta_c + \frac{q_c}{\tau},\end{aligned}$$

however, it is clear that this result diverges at $\tau = 0$, indicating that what is the **cold dark matter velocity isocurvature mode**, $q_c$, must go to zero as it is the only way to ensure that we have physical flow velocities for these three species, thus $\theta_\gamma = \theta_b = \theta_c$. A similar analysis is available to neutrinos, by comparing them to photons: Consider that to lowest order of $k$ in the Poisson equation (eq. (3.15)) for velocity, we obtain, for both photons and neutrinos:

$$\dot{\theta}_{\gamma,\nu} = \frac{k^2}{4}\delta_{\gamma,\nu} + k^2\Psi,$$

where we left out the high order $k^2\sigma_i$ term. This implies that, by the same process we did on baryons and CDM, we have

$$\begin{aligned}\frac{d}{d\tau}(\theta_\gamma - \theta_\nu) &= -\frac{k^2}{4}S_\nu,\\ \theta_c = \theta_b = \theta_\gamma &= \theta_\nu - \frac{k^2\tau}{4}S_\nu - kq_\nu.\end{aligned}$$

where $kq_\nu$ has an extra factor of $k$ to match the scale of the left side, and is known as the **neutrino velocity isocurvature mode**, which is physical and can be used as an initial condition. This tight-coupling approximation leads to a lot of difficulty in numerical calculations, and the methods used to resolve these issues are detailed in section 5.7 of Ma and Bertschinger, 1995.

## 4.2 Adiabatic and Isocurvature Modes

Before we present the traditional solutions to these modes, we should talk briefly about what we mean by "adiabatic" and "isocurvature". An in depth explanation involving a derivation of thermodynamics is available in section 7.4.1 of Piattella, 2018. In this work, we will differentiate these initial conditions by defining their effect on the **gauge invariant curvature perturbations**, $\mathcal{R}$ and $\zeta$, which are defined as

$$\zeta = \Phi + \frac{\sum_J R_J \delta_J}{\sum_J 3(1+w_J)R_J}, \quad \mathcal{R} = \Phi - \frac{\dot{a}}{a}\frac{\sum_J (\rho_J + P_J)\theta_J}{k^2(\rho_{\text{tot}} + P_{\text{tot}})},$$

in the Conformal Newtonian gauge. Expanding these sums and recasting them in terms of our adiabatic and isocurvature modes, we find that in the crossing regime $k\tau \approx 1$, we have:

$$\mathcal{R} = \zeta = C_\gamma + \frac{R_\nu S_\nu + \tilde{R}_m S_m \tau}{4}, \tag{4.6}$$

where we have introduced a shorthand: $R_m = R_b + R_c$, and we suppose some $S_m$ exists such that $R_m S_m = R_b S_b + R_c S_c$, and finally we use the fact that

$$R_m \approx \frac{\rho_m}{\rho_r} = \frac{\Omega_{m,0}}{\Omega_{r,0}} a \approx \frac{\Omega_{m,0} H_0 \tau}{\sqrt{\Omega_{r,0}}}$$

such that if we define $\tilde{R}_m = \Omega_{m,0} H_0 / \sqrt{\Omega_{r,0}}$, then $R_m = \tilde{R}_m \tau$ in radiation domination. If we instead approach this with the goal of obtaining the gauge invariant curvature perturbations in terms of the potentials of the Newtonian gauge, we find that

$$\mathcal{R} = \Phi + \frac{1}{2}(\tau\dot{\Phi} - \Psi), \qquad \zeta = \Phi + \frac{1}{2}(\tau\dot{\Phi} - \Psi) + \frac{1}{6}(k\tau)^2\Phi$$

and together, we can make the observation that in the super-horizon regime, with all of the isocurvature perturbations set to zero, we see

$$\mathcal{R} = \zeta = \Phi - \frac{\Psi}{2} = C_\gamma, \tag{4.7}$$

and now we can notice that if we have adiabatic conditions (that is, no isocurvature perturbations) we **must have curvature**. This defines the distinction between adiabaticity and isocurvature in this approach. From a thermodynamic perspective, this distinction is enumerated by the fact that isocurvature initial conditions contribute entropy, while adiabatic initial conditions do not.

## 4.3 Preparing a General Solution

We now can build out solutions for each mode, which we will do by finding one large general form from which we can just plug in the various initial conditions, obtaining each mode on demand. To do this, we follow the same steps as Piattella, 2018 performs in section 7.3, but in notation consistent with what we have presented so far in this work: Expanding our linearized Einstein equations eqs. (3.10) to (3.13), and evaluating the right side for all species, we obtain

$$\frac{3\dot{a}}{a}\left(\dot{\Phi}-\frac{\dot{a}}{a}\Psi\right)+k^2\Phi = 4\pi G a^2(\rho_\gamma\delta_\gamma+\rho_\nu\delta_\nu+\rho_b\delta_b+\rho_c\delta_c), \tag{4.8}$$

$$k^2\left(\frac{\dot{a}}{a}\Psi-\dot{\Phi}\right) = \frac{4\pi G a^2}{3}(4\rho_\gamma\theta_\gamma+4\rho_\nu\theta_\nu+\rho_b\theta_b+\rho_c\theta_c), \tag{4.9}$$

$$\ddot{\Phi}+\frac{\dot{a}}{a}(2\dot{\Phi}-\dot{\Psi})+\left(\frac{2\ddot{a}}{a}-\left(\frac{\dot{a}}{a}\right)^2\right)\Psi+\frac{k^2}{3}(\Phi+\Psi) = -\frac{4\pi G a^2}{3}(\rho_\gamma\delta_\gamma+\rho_\nu\delta_\nu), \tag{4.10}$$

$$k^2(\Phi+\Psi) = -16\pi G a^2(\rho_\gamma\sigma_\gamma+\rho_\nu\sigma_\nu), \tag{4.11}$$

Applying eq. (4.1), the radiation domination condition ($\dot{a}/a = 1/\tau$, and $R_\gamma + R_\nu \approx 1$), the tight-coupling approximation, and the crossing regime approximation to only keep terms of lowest order in $k\tau$, we see the previous result simplify greatly:

$$2(\tau\dot{\Phi}-\Psi)+4\Phi = 4C_\gamma+R_\nu S_\nu+\tilde{R}_m S_m\tau, \tag{4.12}$$

$$2k^2\tau(\Psi-\tau\dot{\Phi}) = 4\theta_\nu-k^2\tau R_\gamma S_\nu-4kR_\gamma q_\nu, \tag{4.13}$$

$$2\tau^2\ddot{\Phi}+2\tau(2\dot{\Phi}-\dot{\Psi})+2\Psi-4\Phi = -4C_\gamma-R_\nu S_\nu, \tag{4.14}$$

$$(k\tau)^2(\Phi+\Psi) = -6R_\nu\sigma_\nu, \tag{4.15}$$

where the $\sigma_\gamma$ washes out due to the tight-coupling approximation, where the scattering removes any preferred directions for photons. At this point, we have a system that is equivalent to that of Piattella, 2018 (but not identical, due to differences in notation), and we can follow his work in section 7.2.5 to produce our general equation,

$$\tau^2\ddot{\Phi} + 6\tau\dot{\Phi} + 2\left(3 + \frac{4}{5}R_\nu\right)\Phi = 4\left(1 + \frac{3}{5}R_\nu\right)C_\gamma + \frac{2}{5}(1+3R_\nu)R_\nu S_\nu \cdots$$
$$\cdots + \left(\frac{3}{2} + \frac{3}{5}R_\nu\right)\tilde{R}_m S_m \tau - \frac{13}{5}(1-R_\nu)\frac{R_\nu q_\nu}{k\tau}. \tag{4.16}$$

We first investigate the homogeneous part of our solution, setting the right hand side to zero. Presuming we have a solution of the form $\Phi \propto \tau^p$, we are able to find that

$$2p = -5 \pm \sqrt{1 - \frac{32}{5}R_\nu}$$

which correspond to gauge modes (solutions that arise due to choice of gauge, and not the dynamics of the system we want to describe) that diverge at early times. The form of the particular solution is demanded by the right hand side:

$$\Phi = c_1 + c_2\tau + \frac{c_3}{k\tau},$$

whose constants can be identified by plugging this solution into our general equation, obtaining

$$c_1 = \frac{2(5+3R_\nu)}{15+4R_\nu}C_\gamma + \frac{1+3R_\nu}{15+4R_\nu}R_\nu S_\nu, \tag{4.17}$$
$$c_2 = \frac{3(5+2R_\nu)}{8(15+2R_\nu)}\tilde{R}_m S_m, \tag{4.18}$$
$$c_3 = \frac{6(R_\nu - 1)}{(5+4R_\nu)}R_\nu q_\nu. \tag{4.19}$$

From this, we finally have a general solution with which we can evaluate our initial conditions:

$$\Phi = \frac{2(5+3R_\nu)}{15+4R_\nu}C_\gamma + \frac{1+3R_\nu}{15+4R_\nu}R_\nu S_\nu + \frac{3(5+2R_\nu)}{8(15+2R_\nu)}\tilde{R}_m S_m \tau + \frac{6(R_\nu - 1)}{5+4R_\nu}\frac{R_\nu q_\nu}{k\tau}, \tag{4.20}$$
$$\Psi = \frac{2(2R_\nu - 5)}{15+4R_\nu}C_\gamma + \frac{8R_\nu - 11}{2(15+4R_\nu)}R_\nu S_\nu + \frac{5(2R_\nu - 3)}{8(15+2R_\nu)}\tilde{R}_m S_m \tau + \frac{6(R_\nu - 1)}{5+4R_\nu}\frac{R_\nu q_\nu}{k\tau}, \tag{4.21}$$

where we found $\Psi$ by rearranging eq. (4.12) and using our expression for $\Phi$.

### 4.3.1 Adiabatic Mode

We find the adiabatic mode by letting $S_\nu = S_c = S_b = q_\nu = 0$, as mentioned before. Applying this, we see our general solution become

$$\Phi = \frac{2(5+3R_\nu)}{15+4R_\nu} C_\gamma, \tag{4.22}$$

$$\Psi = \frac{2(2R_\nu - 5)}{15+4R_\nu} C_\gamma. \tag{4.23}$$

### 4.3.2 Cold Dark Matter and Baryon Density Isocurvature Modes

CDM and baryons have the same solution in the Newtonian gauge, and they are achieved by letting $S_\nu = q_\nu = C_\gamma = 0$, since our mode has a factor of $\tau$ in our general solution. We know that these modes are isocurvature since our gauge invariant variable exhibits the following property:

$$\zeta = \frac{\tilde{R}_m S_m \tau}{4} \implies \zeta \to 0 \quad \text{as} \quad \tau \to 0,$$

after setting the aforementioned modes to zero. The solutions we obtain in this picture are:

$$\Phi = \frac{3(5+2R_\nu)}{8(15+2R_\nu)} \tilde{R}_m S_m \tau, \tag{4.24}$$

$$\Psi = \frac{5(2R_\nu - 3)}{8(15+2R_\nu)} \tilde{R}_m S_m \tau. \tag{4.25}$$

### 4.3.3 Neutrino Density Isocurvature Mode

To obtain the neutrino density mode, we set $S_m = q_\nu = 0$, and our gauge invariant quantity becomes:

$$\zeta = C_\gamma + \frac{R_\nu S_\nu}{4}.$$

For this mode to be isocurvature, we need $\zeta = 0$, thus $4C_\gamma = -R_\nu S_\nu$, which gives solutions

$$\Phi = \frac{3(R_\nu - 1)}{2(15+4R_\nu)} R_\nu S_\nu, \tag{4.26}$$

$$\Psi = \frac{3(R_\nu - 1)}{15+4R_\nu} R_\nu S_\nu. \tag{4.27}$$

### 4.3.4 Neutrino Velocity Isocurvature Mode

To obtain the neutrino velocity mode, we set $C_\gamma = S_m = S_\nu = 0$, we immediately see the isocurvature condition, $\zeta = 0$, and the potentials become identical:

$$\Psi = \Phi = \frac{6(R_\nu - 1)}{5 + 4R_\nu} \frac{R_\nu q_\nu}{k\tau}. \tag{4.28}$$

5

# Results

In this chapter we give a brief overview of the challenges presented by current literature on the subject, and our approach to resolve some of the differences we observed when researching the topic. We then prepare a solution for the adiabatic and isocurvature modes in the presence of the self-interacting neutrino model derived in Kreisch, Cyr-Racine, and Doré, 2020, and discuss the future avenues for research on these solutions.

## 5.1 Comparing to Literature

Though unplanned, a major part of this thesis became reconciling the notational, logical, and factual differences in our various primary sources (namely Ma and Bertschinger, 1995; Bucher, Moodley, and Turok, 2000; Piattella, 2018; Baumann, 2023). Each of these sources manages to describe primordial cosmological perturbation modes to various extents and degrees of accuracy, but occasionally end up disagreeing on some of the results, with even the gauges being defined differently between each source (see table 5.1). In this work, I ended up using the definition of O. Piatella, even though the method I used for obtaining solutions is, for the most part, identical to the methods of C. Ma and E. Bertschinger. Unfortunately, the topics covered in each of these sources are slightly different, meaning to get a full picture of this topic in a modern perspective, you need to use all of them.

| Author | C. Ma | M. Bucher et. al. | O. F. Piattella | D. Baumann |
|---|---|---|---|---|
| Time-like Potential | $\psi$ | $\phi$ | $\Psi$ | $\Psi$ |
| Space-like Potential | $\phi$ | $\psi$ | $-\Phi$ | $\Phi$ |

Table 5.1: Differences in the stated Conformal Newtonian gauges by primary sources.

This presents a challenge, since older sources like Ma and Bertschinger, 1995 are, for the most part comprehensive, do not cover isocurvature modes, and the work based off them, that of Bucher, Moodley, and Turok, 2000, is riddled with unmentioned definitions and many typos. Piattella, 2018 proves to be a vital source, pointing out and correcting many of the mistakes of Bucher, Moodley, and Turok, 2000, but only derives the isocurvature equations in the Conformal Newtonian gauge, which is problematic when most numerical solvers struggle with some of the pathologies presented by the modes in that gauge. Finally, Baumann, 2023 presents a fantastic overview of the topic, but does not actually derive any of the modes, leaving us without a source to which we may compare Piattella, 2018.

These issues have informed and motivated the structure of this thesis to serve as more than just a derivation of the modes in the presence of self-interacting neutrinos, but to serve as a secondary source that compiles the years of work on this complicated subject in a comprehensive overview that presents a complete derivation of the modes in the traditional perturbed FLRW picture as well as the novel self-interacting picture. We also present a derived equation to move from the Conformal Newtonian gauge to the Synchronous gauge directly within this text, with the hope of having it be integral to comparing the results of sources in the future.

## 5.2 The Self-interacting Neutrino Picture

The main goal of this work has been to prepare the necessary mathematical relations and background, along with a comprehensive picture of the physics involved, to be able to find the solutions for the adiabatic and isocurvature modes in the presence of the novel self-interacting neutrino model developed by Kreisch, Cyr-Racine, and Doré, 2020. To do this we first must consider the effects of this model on our Boltzmann equation. In this work we consider the highly relativistic, early-universe scenario in which neutrinos can be considered approximately massless, and thus the left side of our Boltzmann equation can be left unmodified as long as we remain within this time period. Neutrino self-interaction is inherently conservative of mass-energy, and thus we expect to only see effects to the Boltzmann equation at multipoles $\ell \geq 2$. At these higher multipole values, strongly self-interacting neutrinos effect our Boltzmann equation according to the collision term:

$$C\left[f\right] = \frac{G_{\text{eff}}^2 T_\nu^6}{4} \frac{\partial \ln f}{\partial \ln p} \sum_{\ell=0}^{\infty} (-i)^\ell (2\ell+1) \nu_\ell P_\ell(\mu) \left( A\left(\frac{p}{T_\nu}\right) + B_\ell \left(\frac{p}{T_\nu}\right) - 2D_\ell \left(\frac{p}{T_\nu}\right)\right), \tag{5.1}$$

from Kreisch, Cyr-Racine, and Doré, 2020, in which the values of $\nu_\ell, A, B_\ell$, and $D_\ell$ are found in equations (C43, C52, C53, and C54).

A special discussion is reserved for the $\ell = 2$ case, where the self-interacting neutrinos wash out their own shear-stress due to the fact that these interactions do not prefer a specific direction, and thus any preferred direction that would normally cause a nonzero $\sigma_\nu$ is eliminated and we observe $\sigma_\nu \approx 0$ in the self-interaction picture. This effect persists only in the very hot era of the universe we are considering, and as we get to later times where the universe has had more time to cool down, the self-interactions become weaker and less efficient, at which point $\sigma_\nu \not\approx 0$, and the neutrinos begin to develop a shear stress.

Accounting for these interactions, eq. (3.30) becomes invalid, and we obtain a new solution to our Boltzmann equation in the massless limit (see Kreisch, Cyr-Racine, and Doré, 2020):

$$\frac{\partial F_\ell}{\partial \tau} + k\left(\frac{\ell+1}{2\ell+1}F_{\ell+1} - \frac{\ell}{2\ell+1}F_{\ell-1}\right) - 4\left(\frac{k}{3}\Psi\delta_{\ell 1} - \dot{\Phi}\delta_{\ell 0}\right)\cdots$$
$$\cdots = -\frac{120a}{7\pi^4}G_{\text{eff}}^2 T_\nu^5 F_\ell \int_0^\infty dx\, x^2(A(x) + B_\ell(x) - 2D_\ell(x)), \tag{5.2}$$

where we see that for $\ell = 0, 1$ the right side has $A(x) + B_\ell(x) - 2D_\ell = 0$, and we preserve eqs. (3.28) and (3.29).

With this, we can start from eqs. (4.12) to (4.15), since the fluid equations from the solution to the linearized Einstein equations are still valid. The point where we diverge from the traditional derivation is thus when we set $\sigma_\nu \approx 0$ due to the scattering, and time period we are considering. When we do this, it becomes immediately apparent from eq. (4.15) that:

$$\Phi = -\Psi, \tag{5.3}$$

which we can immediately turn around and apply to eq. (4.12) to obtain a general equation for $\Phi$:

$$2\tau\dot{\Phi} + 6\Phi = 4C_\gamma + R_\nu S_\nu + \tilde{R}_m S_m \tau \tag{5.4}$$

Note that we do not need to include eq. (4.14) since it's first derivative and eq. (4.12)'s second derivative are equal, and thus they represent redundant parts of the solution. The homogeneous solution is trivial, and nonphysical

$$\Phi = C\tau^{-3},$$

since it diverges at $\tau = 0$ for anything with $C \neq 0$. The particular solution can be solved via integration constant, obtaining the result:

$$\Phi = \frac{2}{3}C_\gamma + \frac{1}{6}R_\nu S_\nu + \frac{1}{8}\tilde{R}_m S_m \tau \tag{5.5}$$

note that this solution has *no dependence on the neutrino velocity isocurvature mode*. While it is trivial to show what $\Psi$ is, we can get an equation for $\theta_\nu$ by plugging our result and it's derivative into eq. (4.13), giving:

$$\theta_\nu = \frac{k^2\tau}{4}\left(\left(1 - \frac{4}{3}R_\nu\right)S_\nu - \frac{4}{3}C_\gamma - \frac{1}{4}\tilde{R}_m S_m \tau\right) + (1 - R_\nu)kq_\nu \tag{5.6}$$

Now, we can look at the various modes.

### 5.2.1 Adiabatic Mode

Just like before, we set all isocurvature modes to zero, $S_\nu = S_m = q_\nu = 0$, and investigate our result:

$$\Phi = \frac{2}{3}C_\gamma, \tag{5.7}$$

$$\theta_\nu = \frac{k^2\tau}{3}C_\gamma. \tag{5.8}$$

An important sanity check here is making sure that the adiabaticity condition we showed earlier, eq. (4.7), is satisfied, which it is, since letting $\Phi = -\Psi$ implies that $3\Phi = 2C_\gamma$ which is the exact result we see in our mode.

### 5.2.2 Cold Dark Matter and Baryon Density Isocurvature Modes

We let $C_\gamma = S_\nu = q_\nu = 0$, indicating that once again we have a gauge invariant curvature perturbation that becomes zero at early time $\tau \to 0$. This is sufficient for the isocurvature condition, and we are able to obtain our modes:

$$\Phi = \frac{1}{8}\tilde{R}_m S_m \tau, \tag{5.9}$$

$$\theta_\nu = -\frac{k^2\tau^2}{16}\tilde{R}_m S_m. \tag{5.10}$$

### 5.2.3 Neutrino Density Isocurvature Mode

We let $S_m = q_\nu = 0$, and again have the implication that $4C_\gamma = -R_\nu S_\nu$ since our mode is isocurvature, and thus $\zeta = 0$, which leaves us with

$$\Phi = 0, \tag{5.11}$$

$$\theta_\nu = \frac{k^2 \tau}{4} \left(1 - R_\nu\right) S_\nu. \tag{5.12}$$

### 5.2.4 Neutrino Velocity Isocurvature Mode

We let $C_\gamma = S_\nu = S_m = 0$, observe that $\zeta = 0$ immediately, and obtain

$$\Phi = 0, \tag{5.13}$$

$$\theta_\nu = (1 - R_\nu) k q_\nu. \tag{5.14}$$

# 6
# Conclusions

In this work we have presented a unified picture of the derivation of the primordial perturbation modes under two distinct models: The traditional, flat FLRW model perturbed to the Newtonian gauge, as well as a novel addition to this model in the form of a strongly self-interacting neutrino process, which eliminates the shear-stress of neutrinos and produces some very unique perturbative modes. The purpose of presenting this work as an in depth derivation of the traditional picture stems from the lack of concise, unified literature on the subject, with Piattella, 2018 being the most complete reference for these modes.

Our results in the self-interacting model are very promising, presenting a significant departure from the dynamics we see in the traditional picture. We also see some very exotic effects, such as the gravitational potentials sourcing no scalar perturbations in the case of the neutrino density and velocity isocurvature modes, where we only see scalar perturbations coming from the neutrino velocity.

In presenting a solution to the modes in the presence of self-interacting neutrinos, we have opened up a number of avenues for future research regarding the investigation of the effect of these modes on cosmological observables. The immediate follow up to this work is to convert our results to the Synchronous gauge for comparison and implementation in a Boltzmann solver such as CLASS (Lesgourgues, 2011), or CAMB (Lewis and Challinor, 2011). This work would allow for us to simulate the effect on observables due to our model, allowing us to compare these effects to observational data like PLANCK.

This work also serves as a gateway to the generalization of the quantities we have derived. The presented results are valid under a very specific set of approximations that, in future work, could be progressively relaxed and evaluated for a more complete understanding of the perturbative dynamics of a self-interacting neutrino system, learning about how these modes evolve in later times and different epochs.

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