PHYS 262 - THE WAVE EQUATION, CHAPTER 32

WE WANT TO SHOW THAT MAXWELL'S EQUATIONS ALLOW ELECTROMAGNETIC WAVES. WE DO THIS BY SHOWING THAT $\vec{E}$ AND $\vec{B}$ OBEY THE WAVE EQUATION.

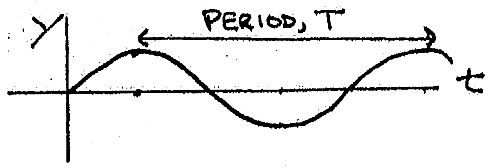
WAVE - PROPAGATION OF ENERGY.

PROPAGATION - OSCILLATION IN BOTH TIME AND SPACE.
↳ PERIODIC MOTION

FOR A MECHANICAL WAVE LIKE SOUND, THE MEDIUM IS OSCILLATING. FOR LIGHT, $\vec{E}$ AND $\vec{B}$ OSCILLATE. SO LIGHT REQUIRES NO MEDIUM, i.e., IT CAN (AND DOES) PROPAGATE THROUGH A VACUUM.

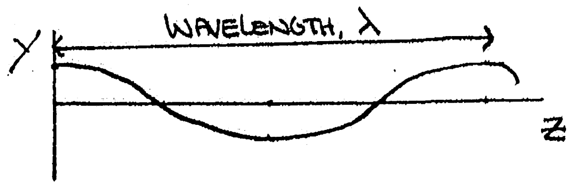
THE SIMPLEST WAVE IS A TRANSVERSE, PERIODIC WAVE ⇒ ONE IN WHICH THE MEDIUM OSCILLATES PERPENDICULAR TO THE PROPAGATION DIRECTION WITH SIMPLE HARMONIC MOTION.

OSCILLATE IN TIME ⇒ EACH POINT UNDERGOES SIMPLE HARMONIC MOTION. LET'S CALL y TO BE THE MEDIUM'S HEIGHT ABOVE ITS EQUILIBRIUM POSITION. FOR A FIXED LOCATION z:



$T = \frac{1}{f}$ f = FREQUENCY

OSCILLATE IN SPACE ⇒ EACH POINT IS NOT IN PHASE WITH THE OTHERS (THEY DON'T HIT THEIR MAXIMA AT THE SAME TIME). IF WE PLOT THE HEIGHTS AT DIFFERENT z's FOR THE SAME TIME, WE GET ANOTHER SINE/COSINE.



$\lambda f = V$ → PROPAGATION SPEED

THE WAVE EQUATION IS THE DIFFERENTIAL EQUATION THAT GIVES y AS A FUNCTION OF BOTH z AND t.

FOR A WAVE PROPAGATING ALONG z WITH A SPEED V, THE WAVE EQUATION IS

$$\frac{\partial^2 y}{\partial z^2} = \frac{1}{V^2}\frac{\partial^2 y}{\partial t^2}$$

MAXWELL'S EQUATIONS SHOW THAT $\vec{E}$ AND $\vec{B}$ BOTH OBEY A WAVE EQUATION. BUT $\vec{E}$ AND $\vec{B}$ ARE 3-D VECTOR QUANTITIES SO THE DERIVATIVE TAKING IS SLIGHTLY MORE COMPLICATED.

MULTI-VARIABLE CALC REVIEW

SCALAR FUNCTIONS: $u = u(x,y,z)$. AT EVERY POINT (x,y,z) u GIVES US A SCALAR. AN EXAMPLE WOULD BE THE TEMPERATURE AT ALL POINTS IN A ROOM.

A DERIVATIVE TELLS US HOW THE FUNCTION IS CHANGING, BUT WE HAVE 3 DIFFERENT DIRECTIONS IN WHICH IT CAN CHANGE. $\Rightarrow$ PARTIAL DERIVATIVE

$\frac{\partial u}{\partial x} \rightarrow$ CHANGE IN u ALONG x KEEPING y AND z FIXED

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EXAMPLE: FIND THE PARTIAL DERIVATIVES OF $u = xyz^2$

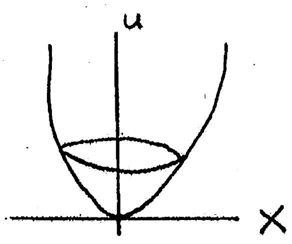
$$\frac{\partial u}{\partial x} = 1(yz^2) = yz^2. \qquad \frac{\partial u}{\partial y} = xz^2. \qquad \frac{\partial u}{\partial z} = xy(2z) = 2xyz$$

THE DIRECTION IN WHICH A SCALAR FUNCTION IS CHANGING THE MOST CAN BE FOUND FROM ITS GRADIENT VECTOR $\vec{\nabla} u$.

$$\boxed{\vec{\nabla} u = \hat{\imath}\frac{\partial u}{\partial x} + \hat{\jmath}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}}$$

$\vec{\nabla} u$ IS A VECTOR WHICH POINTS IN DIRECTION OF GREATEST CHANGE

CONTOUR MAPS OF 3-D FUNCTIONS ARE CREATED BY PLOTTING THE SURFACES WHICH HAVE CONSTANT VALUES OF $|\vec{\nabla} u|$.



EXAMPLE $u = x^2 + y^2 \rightarrow$ 3-D PARABOLA

$$\vec{\nabla} u = 2x\hat{\imath} + 2y\hat{\jmath}$$

AT THE POINT $(x=1, y=1)$ $\vec{\nabla} u = 2\hat{\imath} + 2\hat{\jmath} \Rightarrow$ THE FUNCTION IS CHANGING THE MOST IN A DIRECTION $\phi = \tan^{-1}\left(\frac{2}{2}\right) = 45°$

AT THE POINT $(x=1, y=2)$, $\vec{\nabla} u = 2\hat{\imath} + 4\hat{\jmath} \Rightarrow \phi = \tan^{-1}(4/2) = 63°$

THE CONTOURS OCCUR AT $|\vec{\nabla} u| = \sqrt{(2x)^2 + (2y)^2} = 2\sqrt{x^2+y^2} =$ CONSTANT $\Rightarrow$ CIRCLES

VECTOR FUNCTIONS: $\vec{A} = A_x(x,y,z)\hat{\imath} + A_y(x,y,z)\hat{\jmath} + A_z(x,y,z)\hat{k} \Rightarrow$ COMPONENTS ARE FUNCTIONS

$\vec{E}$ AND $\vec{B}$ ARE VECTOR FUNCTIONS

THERE TWO WAY TO TAKE THE DERIVATIVE OF A VECTOR FUNCTION, ONE GIVES A SCALAR, THE OTHER GIVES A VECTOR.

DIVERGENCE $$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

CURL $$\vec{\nabla} \times \vec{A} = \hat{\imath}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

BOTH COME FROM TREATING $\vec{\nabla}$ AS THE VECTOR $\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

AND USING THE COMPONENT VERSION OF DOT AND CROSS PRODUCT.

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z, \quad \vec{A} \times \vec{B} = \hat{\imath}(A_yB_z - A_zB_y) + \hat{\jmath}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)$$

EXAMPLE: FIND DIVERGENCE AND CURL OF $\vec{A} = \frac{x}{x^2+y^2+z^2}\hat{\imath} + \frac{y}{x^2+y^2+z^2}\hat{\jmath} + \frac{z}{x^2+y^2+z^2}\hat{k}$

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{x^2+y^2+z^2} - \frac{2x(x)}{(x^2+y^2+z^2)^2} + \frac{1}{x^2+y^2+z^2} - \frac{2y(y)}{(x^2+y^2+z^2)^2} + \frac{1}{x^2+y^2+z^2} - \frac{2z(z)}{(x^2+y^2+z^2)^2} = \frac{3}{x^2+y^2+z^2} - \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}$$

$$= \frac{3}{x^2+y^2+z^2} - \frac{2}{x^2+y^2+z^2} = \frac{1}{x^2+y^2+z^2}$$

$$\vec{\nabla}\times\vec{A} = \hat{i}\left(\frac{-2y(z)}{(x^2+y^2+z^2)^2} - \frac{-2z(y)}{(x^2+y^2+z^2)^2}\right) + \hat{j}\left(\frac{-2z(x)}{(x^2+y^2+z^2)^2} - \frac{-2x(z)}{(x^2+y^2+z^2)^2}\right) + \hat{k}\left(\frac{-2x(y)}{(x^2+y^2+z^2)^2} - \frac{-2y(x)}{(x^2+y^2+z^2)^2}\right)$$

$$\Rightarrow \vec{\nabla}\times\vec{A} = 0$$

THE 2ND DERIVATIVE OF A SCALAR FUNCTION IS TAKEN USING THE LAPLACIAN.

LAPLACIAN: $\nabla^2 u = \vec{\nabla}\cdot\vec{\nabla}u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

EXAMPLE: $u = x^2+y^2 \quad \nabla^2 u = 2+2=4$

THE 2ND DERIVATIVE OF A VECTOR FUNCTION MUST BE TAKEN BY COMPONENTS.

WE WRITE: $\nabla^2\vec{A} = \hat{i}(\nabla^2 A_x) + \hat{j}(\nabla^2 A_y) + \hat{k}(\nabla^2 A_z)$

EXAMPLE $\vec{A} = (x^2+y^2)\hat{k} \Rightarrow \nabla^2\vec{A} = \hat{k}(4)$

WE'LL NEED A VERY IMPORTANT RELATIONSHIP BETWEEN DIVERGENCE AND CURL

$$\boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}}$$

EXAMPLE: $\vec{A} = (x^2+y^2)\hat{k} \Rightarrow A_x = 0,\ A_y = 0,\ A_z = x^2+y^2$

$$\vec{\nabla}\times\vec{A} = \hat{i}(2y-0) + \hat{j}(0-2x) + \hat{k}(0-0) = \hat{i}(2y) + \hat{j}(-2x)$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(-2-2) = -4\hat{k}$$

$$\vec{\nabla}\cdot\vec{A} = 0+0+\frac{\partial}{\partial z}(x^2+y^2) = 0 \qquad \nabla^2\vec{A} = \hat{i}(0)+\hat{j}(0)+\hat{k}(4)$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = 0 - 4\hat{k} = -4\hat{k}$$

WE'LL ALSO NEED (WHICH WE'LL GIVE WITHOUT PROOF) TWO FAMOUS THEOREMS

DIVERGENCE THEOREM $\oint \vec{A}\cdot d\vec{a} = \int \vec{\nabla}\cdot\vec{A}\, dV$ (dV = VOLUME ELEMENT

$d\vec{a}$ = AREA ELEMENT

STOKES' THEOREM $\oint \vec{A}\cdot d\vec{\ell} = \int (\vec{\nabla}\times\vec{A})\cdot d\vec{a}$

MAXWELL'S EQUATIONS:

$$\oint \vec{E}\cdot d\vec{A} = Q/\epsilon_0 \qquad \oint \vec{B}\cdot d\vec{\ell} = \mu_0\left(i_c + \epsilon_0 \frac{d\Phi_E}{dt}\right)$$

$$\oint \vec{B}\cdot d\vec{A} = 0 \qquad \oint \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$$

WE USE DIVERGENCE AND STOKES' THMS TO WRITE MAXWELL'S EQUATIONS IN THEIR "DIFFERENTIAL" FORM.

GAUSS'S LAW: $\oint \vec{E}\cdot d\vec{A} = Q/\epsilon_0$. THE DIVERGENCE THEOREM TELLS US

$\oint \vec{E}\cdot d\vec{A} = \int \vec{\nabla}\cdot\vec{E}\, dV$. IF WE WRITE $Q = \int \rho\, dV$ WHERE

ρ = CHARGE DENSITY $\Rightarrow \int \vec{\nabla}\cdot\vec{E}\, dV = \int \rho/\epsilon_0\, dV$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho}$$

G'S LAW FOR MAGNETISM: $\oint \vec{B}\cdot d\vec{A} = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$

FARADAY'S LAW: $\oint \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$

STOKES' THM $\Rightarrow \oint \vec{E}\cdot d\vec{\ell} = \int (\vec{\nabla}\times\vec{E})\cdot d\vec{A}$

$$\Phi_B = \int \vec{B}\cdot d\vec{A} \Rightarrow \frac{d\Phi_B}{dt} = \frac{d}{dt}\int \vec{B}\cdot d\vec{A} = \int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

SO FARADAY'S LAW IS $\int (\vec{\nabla}\times\vec{E})\cdot d\vec{A} = \int \frac{-\partial\vec{B}}{\partial t}\cdot d\vec{A}$ $\left[\frac{\partial\vec{B}}{\partial t} = \hat{\imath}\frac{\partial B_x}{\partial t} + \hat{\jmath}\frac{\partial B_y}{\partial t} + \hat{k}\frac{\partial B_z}{\partial t}\right]$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = \frac{-\partial\vec{B}}{\partial t}}$$

GETS RID OF FLUX.
A CHANGING WITH TIME MAGNETIC FIELD INDUCES AN ELECTRIC FIELD.

AMPERE'S LAW: $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \left(i_c + \epsilon_0 \frac{d\Phi_E}{dt} \right)$

$\oint \vec{B} \cdot d\vec{\ell} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{A}$. $\frac{d\Phi_E}{dt} = \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A}$. $i_c = \int \vec{J} \cdot d\vec{A}$

$\vec{J}$ = CURRENT DENSITY

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)}$$

IN A REGIN OF SPACE WITHOUT CHARGE OR CURRENT (IN A VACUUM AWAY FROM THE ANTENNA) $\rho = 0$, $\vec{J} = 0$, MAXWELL'S EQUATIONS BECOME

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

→ REMEMBER FOR ANY VECTORS: $\vec{F} \times \vec{G} = \vec{H}$, $\vec{H}$ IS PERPENDICULAR TO BOTH $\vec{F}$ AND $\vec{G}$.

$\Rightarrow$ $\vec{E}$ AND $\vec{B}$ MUST BE PERPENDICULAR TO EACH OTHER

USE THE RELATION: $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

$$\Rightarrow \underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{E})}_{-\frac{\partial \vec{B}}{\partial t} \text{ BY FARADAY'S LAW}} = \vec{\nabla} \underbrace{(\vec{\nabla} \cdot \vec{E})}_{\emptyset \text{ BY GAUSS'S LAW}} - \nabla^2 \vec{E}$$

$$\Rightarrow \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\nabla^2 \vec{E} \Rightarrow -\frac{\partial}{\partial t} \underbrace{(\vec{\nabla} \times \vec{B})}_{\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \text{ BY AMPERE'S LAW}} = -\nabla^2 \vec{E}$$

$$\Rightarrow \frac{\partial}{\partial t} \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = +\nabla^2 \vec{E} \Rightarrow \boxed{\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla^2 \vec{E}}$$

LIKEWISE: $\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B}$

$$\Rightarrow \vec{\nabla}\times\left(\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right) = -\nabla^2\vec{B} \Rightarrow \mu_0\epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) = -\nabla^2\vec{B}$$

$$\Rightarrow \mu_0\epsilon_0\left(-\frac{\partial^2\vec{B}}{\partial t^2}\right) = -\nabla^2\vec{B} \Rightarrow \boxed{\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2} = \nabla^2\vec{B}}$$

COMPARE WITH WAVE EQUATION: $\frac{1}{V^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial z^2}$

$\Rightarrow$ FOR ELECTRIC AND MAGNETIC FIELDS, THE PROPAGATION SPEED IS

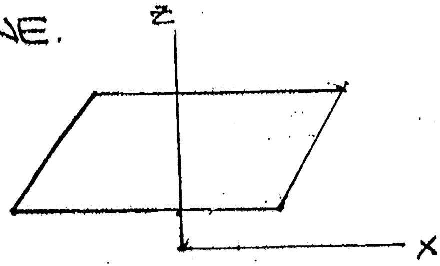
$$V = \frac{1}{\sqrt{\mu_0\epsilon_0}} = \left[\left(4\pi\times10^{-7}\frac{Ns^2}{C^2}\right)\left(8.85\times10^{-12}\,C^2/Nm^2\right)\right]^{-1/2} = 2.999\times10^8\,m/s$$
$$= 3.0\times10^8\,m/s = C$$

MAXWELL WAS THE FIRST PERSON TO <u>CALCULATE</u> THE SPEED OF LIGHT

<u>PLANE WAVES</u> - THE SIMPLEST SOLUTION TO THE WAVE EQUATION(S) WHICH ALSO OBEY MAXWELL'S EQUATIONS HAVE THE FORM

$\vec{E} = \hat{i}\,E(z,t)$ AND $\vec{B} = \hat{j}\,B(z,t)$ WHERE $\hat{k}$ IS THE PROPAGATION DIRECTION. IN OTHER WORDS, $\vec{E}$ AND $\vec{B}$ ARE NOT FUNCTIONS OF X AND Y.

$E(z,t)$ AND $B(z,t)$ ARE CONSTANT ON ANY SURFACE FOR WHICH z IS CONSTANT, i.e., A PLANE.



ALSO: THE DIRECTION OF $\vec{E}\times\vec{B}$ IS $\hat{k}$, THE PROPAGATION DIRECTION

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \rightarrow \frac{\partial^2 E}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \rightarrow \frac{\partial^2 B}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

THE SOLUTION TO THESE EQUATIONS ARE $E = E_0 \cos(Kz - \omega t)$
$B = B_0 \cos(Kz - \omega t)$

$\boxed{K = \frac{2\pi}{\lambda}}$ IS CALLED THE WAVE NUMBER λ = WAVELENGTH

$\boxed{\omega = 2\pi f}$ IS THE ANGULAR FREQUENCY, f = FREQUENCY

$\mu_0 \epsilon_0 = \frac{1}{C^2}$ C = SPEED OF LIGHT $\boxed{\lambda f = c}$

E_0 AND B_0 ARE THE MAXIMUM VALUES, i.e., THE AMPLITUDES FOR E AND B. MAXWELL'S EQUATIONS REQUIRE

$$\boxed{E_0 = c B_0}$$

SO
$$\boxed{\begin{aligned} \vec{E} &= \hat{\imath}\, E_0 \cos(Kz - \omega t) \\ \vec{B} &= \hat{\jmath}\, B_0 \cos(Kz - \omega t) \end{aligned}}$$

PLANE WAVE FIELDS
FOR WAVE PROPAGATING IN +Z, i.e., $\hat{k}$ DIRECTION.