# PHYSICS 262
# FINAL EXAM

Please answer any four of the following five questions. Each question is worth five points. Partial credit will be awarded for any **attempted** problem.

1. Consider the **HELIUM** nucleus that consists of two protons and two neutrons. Since the protons (and neutrons) are confined to the nucleus, the particle in a box problem tells us that the proton energies will be quantized.

   (a) Treating the nucleus as a box whose width $4.8 \times 10^{-15}\,m$ (twice the radius of the helium nucleus), find the ground state energy $E_1$ of the proton. Use $1.00065\,u$ for the mass of the proton, $u = 931.5\,MeV/c^2$ and express your answer in $MeV$. Using $\hbar = 6.583 \times 10^{-22}\,MeV \cdot s$ will prevent the need to convert the mass into kilograms, but you will need to explicitly use $c = 3 \times 10^8\,m/s$.

PARTICLE IN A BOX, $E_n = \dfrac{\pi^2\hbar^2}{2ML^2} n^2 \Rightarrow E_1 = \dfrac{\pi^2\hbar^2}{2ML^2}$

$$\Rightarrow E_1 = \frac{\pi^2(6.583\times10^{-22}\,MeV\cdot s)^2}{2(1.00065)(931.5\,MeV/c^2)(4.8\times10^{-15}\,m)^2}$$

NEED m/s to get correct UNIT.

$$\Rightarrow E_1 = \frac{\pi^2(6.583\times10^{-22}\,MeV\cdot s)^2\,(3\times10^8\,m/s)^2}{2(1.00065)(931.5\,MeV)(4.8\times10^{-15}\,m)^2}$$

$$E_1 = \underline{\underline{8.926\,MeV}}$$

(b) In a "nuclear reaction", the proton may be excited into a higher energy state for brief a period of time. When the proton returns to its ground state, it will emit a photon of light. If the proton is excited to its $n = 3$ state, what frequency of light will be emitted when it returns to the ground state ($n = 1$)? What type of light is this?

$$E_3 = E_1 3^2 = 9E_1$$

$$\Rightarrow \Delta E = E_1 - E_3 = E_1 - 9E_1 = -8E_1$$

$$\Rightarrow E_{photon} = 8E_1 = 8(8.926\,\text{MeV}) = 71.7\,\text{MeV}$$

$$\hbar = h/2\pi \Rightarrow h = 2\pi\hbar$$

$$E_{photon} = hf \Rightarrow f = \frac{71.7\,\text{MeV}}{(6.583\times10^{-22}\,\text{MeV}\cdot\text{s})2\pi}$$

$$\Rightarrow f = \underline{\underline{1.7\times10^{22}\,\text{Hz}}} \leftarrow \text{GAMMA RAY}$$

2. A better, though still very unrealistic, model of the nucleus would be to treat it as a finite well. Here the depth of the well, $U_0$, would come from the "Strong" nuclear force; the fundamental force of nature that cancels the electric repulsion between the positively charged protons in the nucleus.

The strong force's energy comes from the mass of the nucleus's protons and neutrons (protons and neutrons have more mass outside than inside a nucleus). The most common form of helium has two protons and two neutrons, and a mass of $4.002603\,u$ where $u = 931.5\,MeV/c^2$.

(a) Outside of a nucleus, the mass of a proton is $1.00728\,u$ while the neutron's mass is $1.008665\,u$. Find the binding energy per nucleon, $E_b$, in $MeV$ for the helium nucleus. Hint: Find the energy of the "missing" mass and then divide by the total number of protons and neutrons.

$$E_b = \frac{(\Delta M)c^2}{\underset{\text{2 protons + 2 neutrons}}{4}}$$

$$\Delta M = \underset{\text{protons}}{2(1.00728u)} + \underset{\text{Neutrons}}{2(1.008665u)} - \underset{\text{Helium}}{4.002603u}$$

$$= 4.03189u - 4.002603u = .029287u$$

$$= .029287(931.5\,MeV/c^2) = 27.28\,MeV/c^2$$

$$\Rightarrow E_b = \frac{(27.28\,MeV/c^2)c^2}{4} = \frac{27.28\,MeV}{4} = \underline{\underline{6.82\,MeV}}$$

(b) For a finite well of depth $U_0 = E_b$ of part (a) and length $4.8 \times 10^{-15}\,m$ (twice the radius of a helium nucleus), find the transcendental equations for the even and odd wavefunctions' allowed energies in $MeV$ for one of the protons. **Substitute all possible numerical values** in your expressions so that your left with equations depending only the energy, $E$. Assume the proton's mass inside the nucleus is $1.00065\,u$. Using $\hbar = 6.583 \times 10^{-22}\,MeV \cdot s$ will prevent the need to convert the mass into kilograms, but you will need to explicitly use $c = 3 \times 10^8\,m/s$.

Finite well, even: $\tan\left(\frac{kL}{2}\right) = \frac{k'}{k}$, odd: $\cot\left(\frac{kL}{2}\right) = \frac{-k'}{k}$

$$\frac{k'}{k} = \sqrt{\frac{U_0}{E} - 1} = \sqrt{\frac{6.82}{E} - 1}$$

Need m/s to cancel units

$$k = \sqrt{\frac{2mE}{\hbar^2}} = \left[\frac{2(1.00065)(931.5\,MeV/c^2)E}{(6.583\times10^{-22}\,MeV\cdot s)^2}\right]^{1/2}$$

$$\Rightarrow \frac{kL}{2} = \left[\frac{2(1.00065)(931.5)E}{(3\times10^{8})^2(6.583\times10^{-22})^2}\right]^{1/2}\frac{(4.8\times10^{-15})}{2} = 2.186\times10^{14}\sqrt{E}\left(\frac{4.8\times10^{-15}}{2}\right)$$

$$\Rightarrow \frac{kL}{2} = .5247\sqrt{E} \Rightarrow \boxed{\begin{array}{l}\text{Even: } \tan(.5247\sqrt{E}) = \sqrt{\frac{6.82}{E} - 1} \\ \text{Odd: } \cot(.5247\sqrt{E}) = \sqrt{\frac{6.82}{E} - 1}\end{array}}$$

(c) Verify, by checking the numerical values of each side of your equation, that the ground state energy of the proton is 2.[illegible] $MeV$.
2.7913

ground state ⇒ Lowest Energy. Even Solutions give Lowest Energy ⇒ $\tan(.5247\sqrt{E}) = \sqrt{\frac{6.82}{E} - 1}$

$$\tan(\underbrace{.5247\sqrt{2.7913}}_{\text{NO UNIT}}) = \tan(.8766\,rad) = 1.2014 \checkmark$$

$$\sqrt{\frac{6.82}{2.7913} - 1} = \sqrt{1.4433} = 1.2014 \checkmark$$

3. The steady-state wavefunction, $\Phi(x)$ for a particle is given by the following expression, where $\Phi(x)$ has no units when $x$ is in nanometers.

$$\Phi(x) = \begin{cases} C\sqrt{8x - x^4}, & \text{for } 0 \le x \le 2 \\ 0, & \text{else} \end{cases}$$

(a) Find the value of the normalization constant $C$.

Normalized $\Rightarrow \int_{\text{All values}} \Phi^* \Phi \, dx = 1$

$\Phi_{(x)}$ here is real $\Rightarrow \Phi^* \Phi = \Phi^2 = C^2(8x - x^4)$

$0 \le x \le 2$

$$\Rightarrow 1 = \int_0^2 C^2(8x - x^4)\,dx = C^2 \int_0^2 (8x - x^4)\,dx$$

$$= C^2\left(8 \cdot \frac{x^2}{2} - \frac{x^5}{5}\right)\Big|_0^2 = C^2\left(4 \cdot 2^2 - \frac{2^5}{5}\right) = C^2(16 - 6.4)$$

$$\Rightarrow C^2(9.6) = 1 \Rightarrow C = \frac{1}{\sqrt{9.6}} = .3227$$

(b) What is the average value of position, $\bar{x}$, for a particle having this wavefunction?

$$\bar{X} = \int \Phi^* x \, \Phi \, dx = \int_0^2 x \left(C^2 \cdot 8x - x^4\right) dx$$

$$= C^2 \int_0^2 \left(8x^2 - x^5\right) dx = C^2 \left[8 \cdot \frac{x^3}{3} - \frac{x^6}{6}\right] \Big|_0^2$$

$$= C^2 \left(\frac{8}{3} \cdot 2^3 - \frac{1}{6} \cdot 2^6\right) = C^2 (10.6667)$$

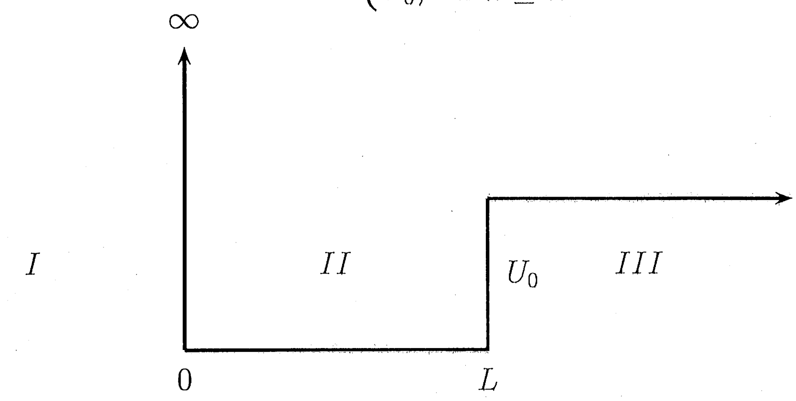
$$= \frac{1}{\sqrt{9.6}} (10.6667) = \underline{\underline{1.11}}$$

→ $\frac{10}{9}$ for FRACTION lovers

← nanometers I suppose

4. The semi-infinite well is the unholy alliance between the particle in a box problem and the finite quantum well. It has the following potential energy.

$$U(x) = \begin{cases} \infty, & \text{if } x = 0 \\ 0, & \text{if } 0 < x < L \\ U_0, & \text{if } x \geq L \end{cases}$$




An electron with energy $E < U_0$ is placed in the semi-infinite well.

(a) Find the general form of the wavefunctions, $\Phi_I$, $\Phi_{II}$, $\Phi_{III}$.

I: No Probability to left $\Rightarrow \boxed{\Phi_I = 0}$

II: Free particle $\Rightarrow \Phi_{II} = Ae^{ikx} + Be^{-ikx}$ $\quad k = \sqrt{\frac{2mE}{\hbar^2}}$

III: classically forbidden $\Rightarrow \Phi_{III} = Ce^{-k'x} + De^{k'x}$ $\quad k' = \sqrt{\frac{2m(U_0-E)}{\hbar^2}}$

(b) What is the boundary condition at $x = 0$? Use this to simplify $\Phi_{II}$.

WAVE FUNCTIONS MUST MATCH $\Rightarrow \Phi_{II}(x=0) = 0$

$\Rightarrow A + B = 0 \Rightarrow B = -A \Rightarrow \Phi_{II} = Ae^{ikx} - Ae^{-ikx}$

$= A(e^{ikx} - e^{-ikx}) = A(2i\sin kx) \Rightarrow$

$$\boxed{\Phi_{II}(x) = F\sin(kx)}$$

(c) Based on physical reasons, which part of $\Phi_{III}$ must be set to zero? (You may have already done this. If so, please re-state your reason why.)

III is $X \geq L \Rightarrow X \to \infty$, $e^{k'x} \to \infty$ AS $X \to \infty$, SO IT IS NONPHYSICAL. $\Rightarrow D = 0$ AND $\therefore$ $\Phi_{III} = Ce^{-k'x}$ ONLY.

(d) Using the simplified $\Phi_{II}$, write down the boundary conditions at $x = L$.

$\Phi_{II} = F\sin kx$, $\Phi_{III} = Ce^{-k'x}$ At $x=L$, $\Phi_{II} = \Phi_{III}$ AND $\frac{d\Phi_{II}}{dx} = \frac{d\Phi_{III}}{dx}$

$\frac{d\Phi_{II}}{dx} = Fk\cos kx$, $\frac{d\Phi_{III}}{dx} = -k'Ce^{-k'x}$

BOUNDARY CONDITIONS : $\boxed{\begin{aligned} &F\sin kL = Ce^{-k'L} \\ &kF\cos kL = -k'Ce^{-k'L} \end{aligned}}$

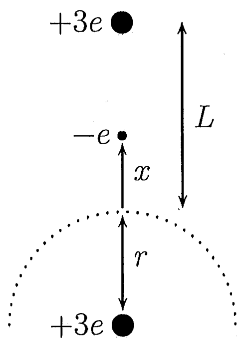
(e) Find the transcendental equation that determines the allowed energies of the electron.

DIVIDE TWO EQUATIONS ABOVE $\Rightarrow \frac{\cancel{F}\sin kL}{k\cancel{F}\cos kL} = \frac{\cancel{C}\cancel{e^{-k'L}}}{-k'\cancel{C}\cancel{e^{-k'L}}}$

$\Rightarrow \frac{\tan kL}{k} = -\frac{1}{k'} \Rightarrow \boxed{\tan kL = \frac{-k}{k'}}$ OR LIKE BEFORE

$\frac{1}{\tan kL} = \frac{-k'}{k} \Rightarrow \boxed{\cot kL = \frac{-k'}{k}}$ $\leftarrow$ ODD, Finite-well transcendental EQN.

5. As an extremely simplified model for the scanning tunneling microscope, consider the problem of two $+3e$ point charges (where $e$ is the proton's charge) separated by a distance $r + L$, where $r$ is the radius atom we are trying to image (in this problem, lithium) and $L$ is the distance between the atom and the microscope's positively charged probe. (We give the probe the same charge as the lithium nucleus to make our calculations easier.)




At a distance $x$ above $r$ is an electron. At that location, we remember that the potential energy of the electron due to the two equal point charges is given by

$$V(x) = -3ke^2\left(\frac{1}{r+x} + \frac{1}{L-x}\right).$$

(a) For $r = .713 \times 10^{-10}\,m$ ($r_2$ for lithium in the Bohr model) and $L = 4 \times 10^{-10}\,m$ (a typical probe distance in an STM experiment), show that the maximum value of the electron's potential energy relative to its starting state is $34.686\,eV$. **Hint:** Find, by the usual calculus means, the maximum of the function $V(x) - V(x=0)$. To help simplify the calculation use the fact that $3ke^2 = 43.152 \times 10^{-10} eV \cdot m$.

$$V(x) - V(x=0) = -3ke^2\left(\frac{1}{r+x} + \frac{1}{L-x}\right) + \underbrace{3ke^2\left(\frac{1}{r} + \frac{1}{L}\right)}_{\text{Constant}}$$

So it suffices to find max of $V(x) \Rightarrow \frac{dV}{dx} = 0$

$$\Rightarrow \frac{d}{dx}\left[-3ke^2\left((r+x)^{-1} + (L-x)^{-1}\right)\right] = -3ke^2\left(\frac{(-1)(1)}{(r+x)^2} + \frac{(-1)(-1)}{(L-x)^2}\right) = 0$$

$$\Rightarrow \frac{-1}{(r+x)^2} + \frac{1}{(L-x)^2} = 0 \Rightarrow \frac{1}{(r+x)^2} = \frac{1}{(L-x)^2} \Rightarrow (L-x)^2 = (r+x)^2 \Rightarrow L-x = r+x$$

$$\Rightarrow x = \frac{L-r}{2}$$

$$V(x=\frac{L-r}{2}) = -3Ke^2\left(\frac{1}{r+\frac{L-r}{2}} + \frac{1}{L-\frac{L-r}{2}}\right) = -3Ke^2\left(\frac{1}{\frac{L}{2}+\frac{r}{2}} + \frac{1}{L/2+r/2}\right)$$

$$= -3Ke^2\left(\frac{2}{L+r} + \frac{2}{L+r}\right) = -3Ke^2\left(\frac{4}{L+r}\right) \Rightarrow V(x) - V(x=0) = 3Ke^2\left(\frac{1}{r} + \frac{1}{L} - \frac{4}{L+r}\right)$$

$$= 43.152\times10^{-10}\,eV\cdot m\left(\frac{1}{.713\times10^{-10}\,m} + \frac{1}{4\times10^{-10}\,m} - \frac{4}{4.713\times10^{-10}\,m}\right) = 43.152\times10^{-10}\,eV\cdot m\,(8.038\times10^{9}\,m^{-1})$$

$$\Rightarrow V(x)-V(x=0)\big|_{max} = 34.685932 \simeq \underline{34.686\,eV}$$

ON THE LAST PAGE IS A ~~plot~~ plot of $V(x)-V(x=0)$. It's FAIRLY BOX-LIKE (KIND OF!)

(b) Find the probability that electron will tunnel from its nucleus to the microscope. Set $U_0 = 34.686\,eV$, $L = 4\times10^{-10}\,m$, and for $E$ use $30.6\,eV$ (the Bohr model's kinetic energy for $n = 2$ lithium). Use $.511\,MeV/c^2$ for the electron mass. **Note:** In this problem, it is ok to use the simplified expression for the tunneling probability equation.

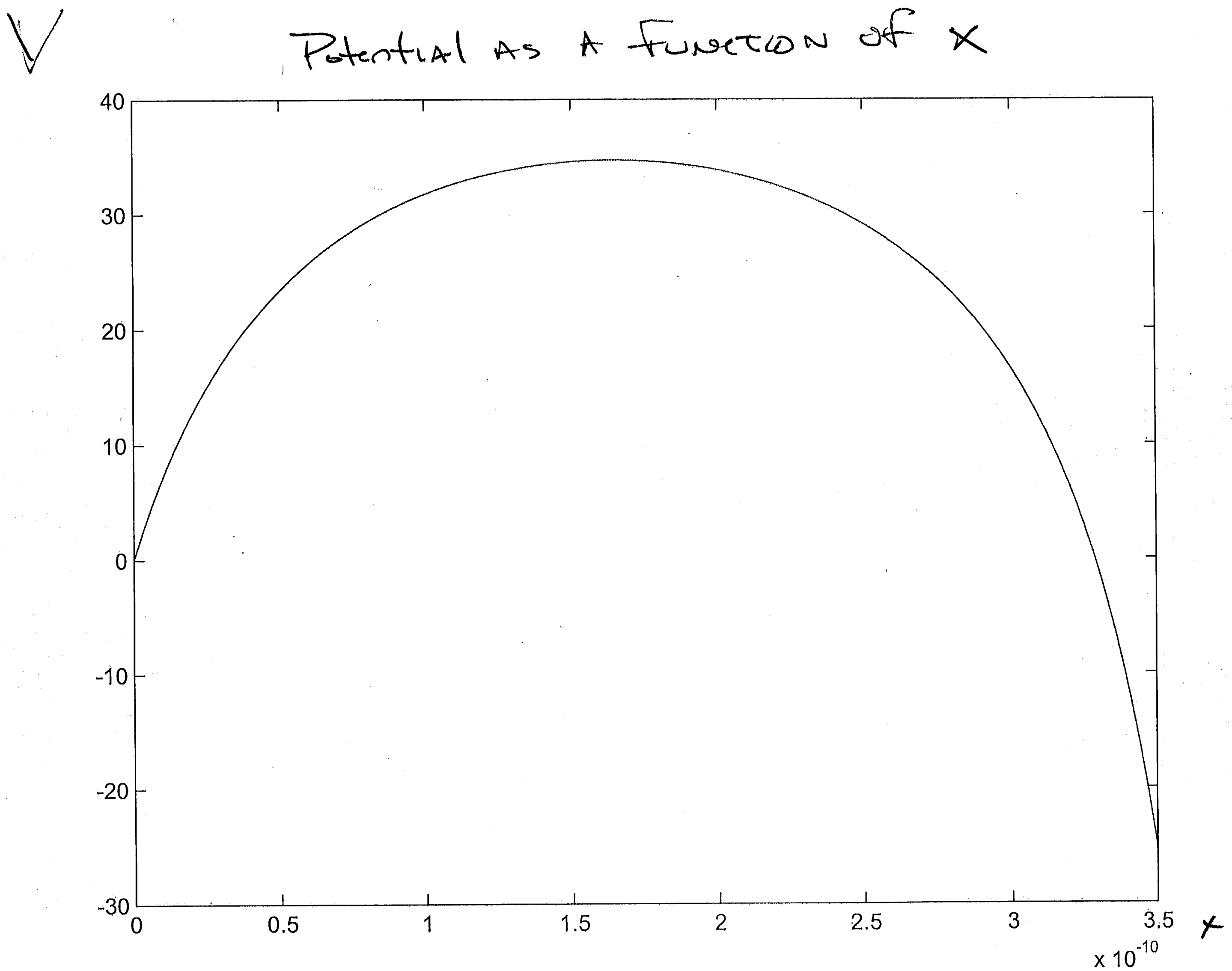
$$T = \frac{16E(U_0-E)}{U_0^2}e^{-2K'L} \qquad E = 30.6eV,\ U_0 = 34.686eV$$
$$\Rightarrow U_0 - E = 4.086eV$$

Need to cancel m & s

$$K' = \sqrt{\frac{2m(U_0-E)}{\hbar^2}} = \left[\frac{2(.511\times10^6\,eV/c^2)(4.086\,eV)}{(6.583\times10^{-16}\,eVs)^2}\right]^{1/2} = \left[\frac{2(.511\times10^6\,eV)(4.086\,eV)}{(3\times10^8\,m/s)^2(6.583\times10^{-16}\,eVs)^2}\right]^{1/2}$$

$$= 1.035\times10^{10}/m \Rightarrow 2K'L = 2(1.035\times10^{10})(4\times10^{-10}) = 8.278$$

$$\Rightarrow T = \frac{16(30.6)(4.086)}{(34.686)^2}e^{-8.278} = \underline{4.22\times10^{-4}}$$


V
Potential as a function of x
40
30
20
10
0
-10
-20
-30
0
0.5
1
1.5
2
2.5
3
3.5
x
x 10^-10