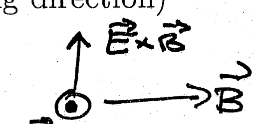
Physics 262
Exam 1

Required Questions

Please answer the following three questions. Each question is worth five points. Partial credit will be awarded for any **attempted** problem.

1. A plane wave with $E_0 = 650\, N/C$ is propagating in the $+y$-direction. If the frequency is $7 \times 10^{14}\, Hz$, write the equations (including direction) for the plane wave's electric and magnetic fields.



propagating in $+y \Rightarrow \vec{E} = \hat{k} E_0 \cos(ky - \omega t)$, $\vec{B} = \hat{i} B_0 \cos(ky - \omega t)$

$E_0 = 650 N/C$, $B_0 = E_0/c = \frac{650 N/C}{3\times10^8 m/s} = 2.17\times10^{-6} T$, $\omega = 2\pi f = 2\pi(7\times10^{14} Hz) = 4.4\times10^{15}$ rad/s

$\lambda = \frac{c}{f} = \frac{3\times10^8 m/s}{7\times10^{14}/s} = 4.29\times10^{-7} m$, $K = \frac{2\pi}{\lambda} = \frac{2\pi}{4.29\times10^{-7} m} = 1.47\times10^{7} rad/m$

$\Rightarrow \vec{E} = \hat{k}\, 650 \cos(1.47\times10^{7} y - 4.4\times10^{15} t)$

$\vec{B} = \hat{i}\, (2.17\times10^{-6}) \cos(1.47\times10^{7} y - 4.4\times10^{15} t)$

2. A $50\, mW$ (mili-Watt) green laser pointer, $\lambda = 532\, nm$, creates a $10\, mm$-radius circular dot. What is the amplitude of the electric and magnetic fields assuming the laser creates a plane wave and that all of the laser's power is in the plane wave?

$I = \frac{1}{2\mu_0}(E_0 B_0)$, $B_0 = E_0/c \Rightarrow I = \frac{1}{2\mu_0}\left(E_0^2/c\right) = \frac{1}{2\mu_0} E_0^2 \left(\sqrt{\mu_0 \epsilon_0}\right)$

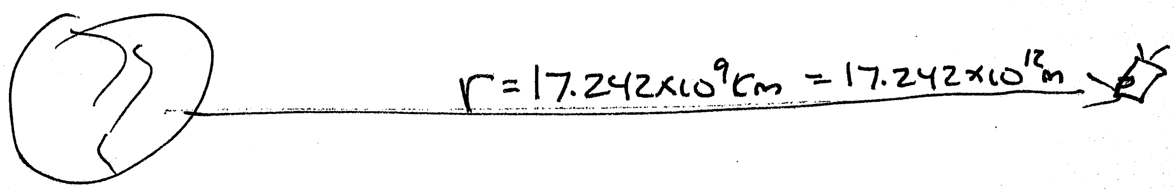
$= \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\, E_0^2$ $\quad I = \frac{Power}{Area} = \frac{50\times10^{-3} Watt}{\underbrace{\pi(10\times10^{-3} m)^2}_{circle}} = 159.15 Watt/m^2$

$\Rightarrow 159.15 Watt/m^2 = \frac{1}{2}\sqrt{\frac{8.85\times10^{-12} C^2/N\cdot m^2}{4\pi\times10^{-7} T\cdot m/A}}\, E_0^2 \Rightarrow E_0 = 346 N/C$

$\Rightarrow B_0 = 1.15\times10^{-6} T$

3. On January 23, 2011, the Voyager I spacecraft was $17.242 \times 10^9\, km$ from Earth. How long (in hours) does it take a radio signal from Voyager I to reach Earth?
Voyager I is fitted with a $23\, Watt$ radio transmitter. Assuming the dispersion of Voyager I's emitted radio waves is only 1×10^{-6}, what is the intensity of the signal when it reaches earth? **HINT:** Let's define the dispersion to be the fraction of the entire $4\pi r^2$ surface area over which the signal is transmitted.

Not to scale!



$$t = \frac{r}{c} = \frac{17.242\times10^{12}\,m}{3\times10^{8}\,m/s} = 57473\,s = 15.96\,hour = \underline{16\ hours!}$$

$$\underline{I} = \frac{P}{A} = \frac{P}{(1\times10^{-6})4\pi r^2} = \frac{23\,Watt}{(1\times10^{-6})4\pi(17.242\times10^{12}\,m)^2}$$

$$\Rightarrow \underline{I} = 6.16\times10^{-21}\,Watt/m^2$$

I'm impressed with NASA!!

Challenge Questions

Please answer **ONE** of the following two questions. This question is worth ten points.

1. Find the charge density, ρ, associated with the following electric field equation where both ρ_0 and r_0 are constant.

$$\vec{\mathbf{E}} = \frac{\rho_0}{4r_0\epsilon_0}\sqrt{x^2+y^2+z^2}\left(x\hat{\imath}+y\hat{\jmath}+z\hat{\boldsymbol{k}}\right)$$

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \Rightarrow \rho = \epsilon_0\,\vec{\nabla}\cdot\vec{E} = \epsilon_0\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right)$$

$$\vec{E} = \frac{\rho_0}{4r_0\epsilon_0}\sqrt{x^2+y^2+z^2}\,(x\hat{\imath}+y\hat{\jmath}+z\hat{k})$$

$$\Rightarrow E_x = \frac{\rho_0}{4r_0\epsilon_0}\left(\sqrt{x^2+y^2+z^2}\right)x,\quad E_y = \frac{\rho_0}{4r_0\epsilon_0}\left(\sqrt{x^2+y^2+z^2}\right)y,\quad E_z = \frac{\rho_0}{4r_0\epsilon_0}\left(\sqrt{x^2+y^2+z^2}\right)z$$

$$\Rightarrow \frac{\partial E_x}{\partial x} = \frac{\rho_0}{4r_0\epsilon_0}\left[\frac{\frac{1}{2}(2x)}{\sqrt{x^2+y^2+z^2}}\,x + \sqrt{x^2+y^2+z^2}\right] = \frac{\rho_0}{4r_0\epsilon_0}\left[\frac{x^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right]$$

Likewise $$\frac{\partial E_y}{\partial y} = \frac{\rho_0}{4r_0\epsilon_0}\left[\frac{y^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right],\quad \frac{\partial E_z}{\partial z} = \frac{\rho_0}{4r_0\epsilon_0}\left[\frac{z^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right]$$

$$\Rightarrow \epsilon_0\,\vec{\nabla}\cdot\vec{E} = \frac{\rho_0}{4r_0}\left[\frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} + 3\sqrt{x^2+y^2+z^2}\right] = \frac{\rho_0}{4r_0}\left[\sqrt{x^2+y^2+z^2}+3\sqrt{x^2+y^2+z^2}\right]$$

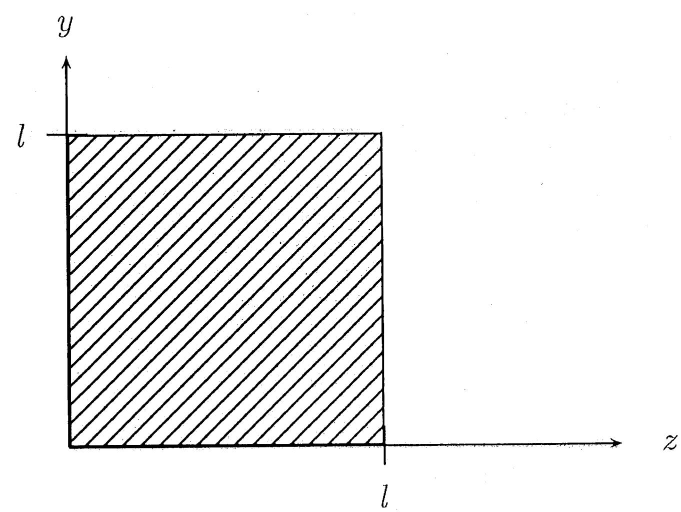
$$= \frac{\rho_0}{4r_0}\left[4\sqrt{x^2+y^2+z^2}\right] = \frac{\rho_0}{r_0}\sqrt{x^2+y^2+z} = \frac{\rho_0 r}{r_0}.$$ where $r = \sqrt{x^2+y^2+z^2}$

2. A *very* simple antenna is a loop of wire. Changing magnetic flux (provided by our wave) will induce a voltage in the coil. The induced voltage in the coil, ϵ, is given by Faraday's law (which regrettably will have to be the integral version here) $\epsilon = -\frac{d\Phi_B}{dt}$.

Assume we have a plane wave with amplitude $B_0 = 1 \times 10^{-6}\,T$ and frequency $f = 1.0 \times 10^{9}\,Hz$. Moreover, our antenna is oriented for maximum flux, *i.e.*, the flux is simply given by

$$\Phi_B = \int B\,dz\,dy$$

where $B = B_0 \cos(kz - \omega t)$. Finally, let's make our integration as easy as possible by assuming our loop is a square in the $z - y$ plane with sides of length $l = 3\,m$. So we need to integrate over the following area:



Find the loop's induced voltage as a function of time t. What is the maximum induced voltage?

$$\Phi_B = \int_0^l\int_0^l B_0 \cos(kz-\omega t)\,dz\,dy = B_0\int_0^l dy\left[\int_0^l \cos(kz-\omega t)\,dz\right]$$

$$= B_0\int_0^l dy\left[\frac{1}{k}\sin(kz-\omega t)\Big|_{z=0}^{l}\right] = B_0\int_0^l dy\,\frac{1}{k}\left[\sin(kl-\omega t)-\sin(-\omega t)\right]$$

$$= B_0\int_0^l dy\,\frac{1}{k}\left[\underbrace{\sin(kl-\omega t)+\sin\omega t}_{\text{No } y\text{-dependence}}\right] = \frac{B_0}{k}\left[\sin(kl-\omega t)+\sin\omega t\right]\int_0^l dy$$

$$\Phi_B = \frac{B_0}{K}\left[\sin(K\ell - \omega t) + \sin\omega t\right]K$$

$$= \frac{B_0 \ell}{K}\left[\sin(K\ell - \omega t) + \sin\omega t\right]$$

$$\mathcal{E}_{ind} = -\frac{d\Phi_B}{dt} = -\frac{B_0\ell}{K}\left[\cos(K\ell - \omega t)(-\omega) + \cos(\omega t)(\omega)\right]$$

$$= \frac{B_0\ell\omega}{K}\left[\cos(K\ell - \omega t) - \cos\omega t\right]$$

$$K = \frac{2\pi}{\lambda}, \quad \omega = 2\pi f \Rightarrow \frac{\omega}{K} = \frac{2\pi f}{2\pi/\lambda} = \lambda f = c$$

$$\Rightarrow \boxed{\mathcal{E}_{IND} = B_0\ell c\left[\cos(K\ell - \omega t) - \cos\omega t\right]}$$

$$B_0 = 1\times10^{-6}\,\mathrm{T},\ \ell = 3\,\mathrm{m},\ c = 3\times10^{8}\,\mathrm{m/s} \Rightarrow \boxed{B_0\ell c = 900\,\mathrm{V}}$$

This is what I had in mind for the Answer.

<u>But</u> I did something stupid.

$$\frac{\omega}{K} = c \Rightarrow K = \frac{\omega}{c} = \frac{2\pi(1\times10^{9})}{3\times10^{8}} = 2\pi\left(\tfrac{1}{3}\times10\right), \Rightarrow K\ell = K(3) = 2\pi(10) = 20\pi$$

$$\Rightarrow \mathcal{E}_{IND} = B_0\ell c\left[\cos(20\pi - \omega t) - \cos\omega t\right], \quad \cos(20\pi - \omega t) = \cos(-\omega t) = \cos\omega t$$

$$\Rightarrow \mathcal{E}_{IND} = B_0\ell c\left[\cos\omega t - \cos\omega t\right] = 0!$$ oops.