16.2 ⋆⋆ The equation of motion of the ith mass is

$$m\ddot{u}_i = T(\sin\theta_i - \sin\theta_{i-1}) = \left(\frac{u_{i+1}-u_i}{b} - \frac{u_i - u_{i-1}}{b}\right) \tag{i}$$

where m is the mass of any one of the masses, dots denote differentiation with respect to t, and θ_i is the angle between the horizontal and the string to the right of the ith mass. If now we let $b \to 0$ holding μ constant, then we can replace m by $m = \mu b$, and u_i by $u_i = u(x)$ [actually $u(x,t)$ but I'll omit the t], and $(u_i - u_{i-1})/b \to u'(x)$, where a prime denotes differentiation with respect to x. With these replacements, Eq.(i) becomes (after dividing by b)

$$\mu\ddot{u}(x) = T\,\frac{u'(x) - u'(x-b)}{b} \to Tu''(x), \quad \text{as } b \to 0$$

which is the wave equation.

16.4 ⋆ If we make the suggested substitutions, then $x = (\xi + \eta)/2$ and $t = (\eta - \xi)/2c$. Therefore,

$$\frac{\partial}{\partial\xi} = \frac{\partial x}{\partial\xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial\xi}\frac{\partial}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2c}\frac{\partial}{\partial t}$$

and

$$\frac{\partial}{\partial\eta} = \frac{\partial x}{\partial\eta}\frac{\partial}{\partial x} + \frac{\partial t}{\partial\eta}\frac{\partial}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2c}\frac{\partial}{\partial t}.$$

Therefore

$$\frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta} = \frac{1}{4c^2}\left(c\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)\left(c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) = \frac{1}{4c^2}\left(c^2\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)$$

which is the claimed identity.

16.9 ⋆⋆ **(a)** We are certainly free to try for a solution of the form $u(x,t) = X(x)T(t)$. If we substitute this into Eq.(16.19), $\partial^2 u/\partial t^2 = c^2\partial^2 u/\partial x^2$, we find $X(x)T''(t) = c^2 T(t)X''(x)$, from which it follows that

$$\frac{T''(t)}{T(t)} = c^2\frac{X''(x)}{X(x)} \tag{ii}$$

where, as usual, primes indicate differentiation with respect to the argument. This equation must hold for all t and all x in the ranges of interest. If we temporarily fix x, then Eq.(ii) says that the left side is independent of t. Similarly, if we fix t, we see that the right side is independent of x. Since the two sides are equal, this implies that both sides are equal to a constant K, independent of t and x. As discussed in the footnote, this constant has to be negative, so we can call it $-\omega^2$. Therefore, $T''(t) = -\omega^2 T(t)$, which implies that $T(t) = A\cos(\omega t - \delta)$ and hence that $u(x,t)$ has the form (16.21).

16.12 $\star\star$ **(a)** Since $f(\xi)$ is localized around $\xi = 0$, the function $f(x+ct)$ is localized around $x = -ct$. When $t = t_\text{o}$, with t_o large and negative, this means that $f(x+ct)$ is localized far to the right and traveling in toward the origin.

(b) Consider the wave $u(x,t) = f(x+ct) - f(-x+ct)$. The second term is called the "image" because it is the result of reflecting the first term in the plane $x = 0$ (and inverting it). Because both terms satisfy the wave equation, so does $u(x,t)$. When t is large and negative, the second term in $u(x,t)$ is zero everywhere that $x \geq 0$ (that is, where the string is actually located). Therefore, $u(x,t)$ is exactly equal to the wave of part (a) everywhere on the string. If we put $x = 0$, we find $u(0,t) = 0$.

(c) Because $u(x,t)$ satisfies the wave equation, the initial conditions, and the boundary conditions, it is the solution. As long as t remains large and negative, the second term is zero (where the string is), so the wave continues to move in from the right toward $x = 0$. As t nears 0, the "image" wave begins to emerge onto the string, and the two terms interfere. Once t is large and positive, the original wave has disappeared into the region $x < 0$, and we are left with just the reflected "image" wave traveling to the right.

16.16 $\star\star$ **(a)** We wish to prove that, if f is spherically symmetric, $f = f(r)$, then

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf). \tag{iv}$$

We'll look at the two sides of this equation in turn. First,

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(f'(r)\frac{\partial r}{\partial x}\right) &= f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\frac{\partial^2 r}{\partial x^2} \\ &= f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r}\left(1 - \frac{x^2}{r^2}\right)\end{aligned} \tag{v}$$

where in pasing to the second line I used that $\partial r/\partial x = x/r$ [first derived in (4.42)] and hence $\partial^2 r/\partial x^2 = (1 - x^2/r^2)/r$. Adding (v) to the two corresponding results in y and z, we find that

$$\nabla^2 f = f''(r) + \frac{2f'(r)}{r}. \tag{vi}$$

Meanwhile the right side of Eq(iv) is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) = \frac{1}{r}\frac{\partial}{\partial r}(rf' + f) = \frac{1}{r}(rf'' + 2f').$$

This is the same as (vi), and we've proved (iv).

(b) The formula inside the back cover for ∇^2 in spherical polars contains three terms, but two of them involve derivatives with respect to θ or ϕ. Acting on $f(r)$ these two give zero, so we're left with a single term that is precisely the right side of Eq.(iv).

16.20 $\star\star$ The given surface has the form $f = x^2 + y^2 + 2z^2 = 4$. The normal to this surface is in the direction of $\nabla f = (2x, 2y, 4z) = 2(1,1,2)$ at the point P with coordinates $(1,1,1)$. Thus the unit normal at P is $\mathbf{n} = (1,1,2)/\sqrt{6}$. Written as matrices, $\mathbf{\Sigma}$ (evaluated at P), $\mathbf{n}$, and $\mathbf{F} = \mathbf{\Sigma}\,\mathbf{n}\,dA$ are as follows:

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{n} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{F} = \mathbf{\Sigma}\,\mathbf{n}\,dA = \frac{dA}{\sqrt{6}}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}.$$

16.31 ⋆ The times of travel of the two waves are $t_{\text{long}} = d/c_{\text{long}}$ and $t_{\text{tran}} = d/c_{\text{tran}}$, where d is the distance from the quake to the observer. Therefore the time between the arrivals of the two signals is $\Delta t = t_{\text{tran}} - t_{\text{long}} = d(1/c_{\text{tran}} - 1/c_{\text{long}})$, and

$$d = \frac{\Delta t}{1/c_{\text{tran}} - 1/c_{\text{long}}} = \frac{720 \text{ s}}{(1/3.0 - 1/5.25) \text{ s/km}} = 5040 \text{ km}.$$

16.37 ⋆⋆ The force exerted by a sliver of air of area dA on the air just ahead of it is $\mathbf{F} = p\mathbf{n}dA = (p_{\text{o}}+p')\mathbf{n}dA$ where $\mathbf{n}$ is the normal to the sliver, in the direction of propagation. The rate at which this force does work is $\mathbf{F}\cdot\mathbf{v}$ and the intensity I is found by dividing this by the area, $I = \mathbf{F}\cdot\mathbf{v}/dA = (p_{\text{o}} + p')\mathbf{n}\cdot\mathbf{v}$. According to (16.142), $\mathbf{v} = p'\mathbf{n}/c\varrho_{\text{o}}$. Therefore $I = (p_{\text{o}} + p')p'/c\varrho_{\text{o}}$. This gives I as the sum of two terms. When p' varies sinusoidally, the first term averages to zero, and we're left with $\langle I\rangle = \langle p'^2\rangle/c\varrho_{\text{o}}$, as claimed.
