13.2 ⋆ The Lagrangian is $\mathcal{L} = \frac{1}{2}m\dot{x}^2 + mgx$, so $p = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$, and $\mathcal{H} = p\dot{x} - \mathcal{L} = p^2/2m - mgx$.The Hamilton equations are

$$\dot{x} = \partial\mathcal{H}/\partial p = p/m \quad \text{and} \quad \dot{p} = -\partial\mathcal{H}/\partial x = mg.$$

Combining the two Hamilton equations, we find that $\ddot{x} = g$ as expected.

13.3 ⋆ The moment of inertia of a uniform disc is $I = \frac{1}{2}MR^2$, and its kinetic energy is $\frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{x}^2/R^2$. Therefore, $\mathcal{L} = \frac{1}{2}(m_1+m_2+\frac{1}{2}M)\dot{x}^2+(m_1-m_2)gx$, $p = (m_1+m_2+\frac{1}{2}M)\dot{x}$, and $\mathcal{H} = p\dot{x} - \mathcal{L} = p^2/2(m_1+m_2+\frac{1}{2}M) - (m_1-m_2)gx$. The Hamilton equations are

$$\dot{x} = \partial\mathcal{H}/\partial p = p/(m_1+m_2+\tfrac{1}{2}M) \quad \text{and} \quad \dot{p} = -\partial\mathcal{H}/\partial x = (m_1-m_2)g,$$

and the acceleration is $\ddot{x} = g(m_1-m_2)/(m_1+m_2+\frac{1}{2}M)$.

13.4 ⋆ The two original coordinates are x and y, and the constraint equation is $x + y + \pi R$ = const. Thus the equations for x and y in terms of the generalized coordinate x are $x = x$ (of course) and $y = -x +$ const, both of which are independent of time.

13.7 ⋆⋆⋆ **(a)** The height of the track is $y = h(x)$. Therefore the distance traveled by the car in a small displacement is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + h'(x)^2}dx. \qquad \text{(i)}$$

It follows that the car's speed satisfies $v^2 = (1 + h'^2)\dot{x}^2$, so the Lagrangian and generalized momentum are

$$\mathcal{L} = \tfrac{1}{2}m\dot{x}^2[1 + h'(x)^2] - mgh(x) \quad \text{and} \quad p = \frac{\partial\mathcal{L}}{\partial\dot{x}} = m\dot{x}[1 + h'(x)^2]. \qquad \text{(ii)}$$

The second equation is easily solved for $\dot{x}$ and the Hamiltonian is

$$\mathcal{H} = \dot{x}p - \mathcal{L} = \frac{p^2}{2m(1 + h'^2)} + mgh$$

(b) Hamilton's equations are

$$\dot{x} = \frac{\partial\mathcal{H}}{\partial p} = \frac{p}{m(1 + h'^2)} \quad \text{and} \quad \dot{p} = -\frac{\partial\mathcal{H}}{\partial x} = \frac{p^2h'h''}{m(1 + h'^2)^2} - mgh' = m(\dot{x}^2h'h'' - gh') \quad \text{(iii)}$$

where in the last step I used Eq.(ii) to replace p by $p = m\dot{x}(1 + h'^2)$.

Before we do anything with this Hamiltonian result, let us look at the Newtonian prediction,

$$m\ddot{s} = F_{\text{tang}} = -dU/ds = -mgh'/\sqrt{1 + h'^2}, \qquad \text{(iv)}$$

where, in the last step, I wrote $U = mgh$ and I used Eq.(i) to replace ds by $dx\sqrt{1 + h'^2}$. To replace $\ddot{s}$ by $\ddot{x}$, note that

$$\ddot{s} = \frac{d}{dt}\frac{ds}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\sqrt{1 + h'^2}\right) = \ddot{x}\sqrt{1 + h'^2} + \frac{h'h''\dot{x}^2}{\sqrt{1 + h'^2}}.$$

Inserting this result into (iv) and solving for $\ddot{x}$ we find that, according to Newton,

$$\ddot{x} = -\frac{gh' + h'h''\dot{x}^2}{1 + h'^2}. \tag{v}$$

Let's now see that we get the same result from the Hamilton equations (iii). From the first of Eqs.(iii) we find

$$\ddot{x} = \frac{d}{dt}\dot{x} = \frac{d}{dt}\frac{p}{m(1+h'^2)} = \frac{\dot{p}}{m(1+h'^2)} - \frac{p}{m}\frac{2h'h''\dot{x}}{(1+h'^2)^2}$$

If we use the second Hamilton equation (iii) to eliminate $\dot{p}$ and the second of Eqs.(ii) to eliminate p, this is easily seen to be exactly the same as the Newtonian result (v).

There is a much simpler way to accomplish the same result, though it may seem a cheat at first sight. Hamilton's equations, like Lagrange's from which we derived them, are true with respect to any choice of generalized coordinates. Therefore we can handle the same problem using as our generalized coordinate s, the distance measured along the track. If we do this, then the Lagrangian is $\mathcal{L} = \frac{1}{2}m\dot{s}^2 - U(s)$ and the generalized momentum is $p = \partial\mathcal{L}/\partial s = m\dot{s}$. Thus the Hamiltonian is $\mathcal{H} = p^2/(2m) + U(s)$ and the second Hamilton equation is $\dot{p} = -\partial\mathcal{H}/\partial s = -dU/ds$ or $m\ddot{s} = -dU/ds$ in agreement with the Newtonian result Eq.(iv).

13.8 ⋆ Since $U = 0$, $\mathcal{L} = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Therefore, $p_x = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$, and similarly $p_y = m\dot{y}$ and $p_z = m\dot{z}$, and finally $\mathcal{H} = \mathbf{p}\cdot\dot{\mathbf{r}} - \mathcal{L} = \mathbf{p}^2/2m$. The six Hamilton equations are

$$\dot{x} = \frac{\partial\mathcal{H}}{\partial p_x} = \frac{p_x}{m} \quad \text{and} \quad \dot{p_x} = -\frac{\partial\mathcal{H}}{\partial x} = 0$$

with similar equations for the y and z components. We can combine these into two vector equations $\dot{\mathbf{r}} = \mathbf{p}/m$ and $\dot{\mathbf{p}} = 0$, with the expected solutions $\mathbf{p} = \text{const} = \mathbf{p}_\text{o}$ and $\mathbf{r} = \mathbf{r}_\text{o} + \mathbf{v}_\text{o}t$ where $\mathbf{v}_\text{o} = \mathbf{p}_\text{o}/m$.

13.10 ⋆ The KE is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$. If we choose the PE to be zero at the origin, $U = -\int_0^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}kx^2 - Ky$. The generalized momenta are given by $\mathbf{p} = m\dot{\mathbf{r}}$ and the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}(p_x^2 + p_y^2) + \tfrac{1}{2}kx^2 - Ky.$$

The two Hamilton equations for x are

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m} \quad \text{and} \quad \dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = -kx$$

which combine to give $\ddot{x} = -(k/m)x$. Thus x oscillates in SHM, $x = A\cos(\omega t - \delta)$, with angular frequency $\omega = \sqrt{k/m}$. Meanwhile, the two y equations are

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{p_y}{m} \quad \text{and} \quad \dot{p}_y = -\frac{\partial \mathcal{H}}{\partial y} = K$$

which combine to give $\ddot{y} = K/m$. Thus y accelerates in the positive y direction, $y = \frac{1}{2}(K/m)t^2 + v_{y\text{o}}t + y_\text{o}$, with constant acceleration K/m.

13.12 ⋆ As generalized coordinate I'll use the bead's position x relative to the axis of spin, as measured in the frame of the rod. The bead's PE is zero and its KE (relative to the ground) is $T = \frac{1}{2}m(\dot{x}^2 + x^2\omega^2)$, so $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + x^2\omega^2)$ and the generalized momentum is $p = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$. Therefore the Hamiltonian is

$$\mathcal{H} = p\dot{x} - \mathcal{L} = \frac{p^2}{2m} - \tfrac{1}{2}mx^2\omega^2.$$

This is not equal to the energy $T+U$ (neither relative to the rod nor relative to the ground), because

$$(T+U)_{\text{(rel to rod)}} = \tfrac{1}{2}m\dot{x}^2 = \frac{p^2}{2m} \neq \mathcal{H}.$$

and

$$(T+U)_{\text{(rel to ground)}} = \tfrac{1}{2}m(\dot{x}^2 + x^2\omega^2) = \frac{p^2}{2m} + \tfrac{1}{2}mx^2\omega^2 \neq \mathcal{H}.$$

13.17 ⋆⋆⋆ **(a)** From (13.32) we see that $\dot{z} = 0$ if and only if $p_z = 0$, and from (13.34) that this implies that

$$\frac{p_\phi^2}{mc^2 z_\text{o}^3} = mg \quad \text{or} \quad z_\text{o} = \left(\frac{p_\phi^2}{m^2c^2g}\right)^{1/3}.$$

(b) If we combine the two equation (13.34) to give $\ddot{z}$ and then put $z = z_\text{o} + \epsilon$, we find

$$\ddot{\epsilon} = \ddot{z} = \frac{\dot{p}_z}{m(c^2+1)} = \frac{1}{m(c^2+1)}\left[\frac{p_\phi^2}{mc^2z^3} - mg\right]$$
$$= \frac{1}{m(c^2+1)}\left[\frac{p_\phi^2}{mc^2z_o^3}\left(1 - 3\frac{\epsilon}{z_o}\right) - mg\right] = -\frac{3p_\phi^2\epsilon}{m^2c^2(c^2+1)z_o^4}$$

where in passing to the second line I used the binomial approximation for z^{-3} and where, in the first expression of the second line, the first and last terms cancelled because of the result of part (a). This equation implies that ϵ oscillates in SHM.

(c) The last equation of part (b) implies that the frequency of these oscillations is

$$\omega = \frac{p_\phi}{mcz_o^2}\sqrt{\frac{3}{c^2+1}} = \sqrt{3}\dot{\phi}_o\frac{c}{\sqrt{c^2+1}} = \sqrt{3}\dot{\phi}_o \sin\alpha$$

where for the second equality I used (13.32) to replace p_ϕ by $mc^2z_o^2\dot{\phi}_o$ and for the third I replaced $c/\sqrt{c^2+1}$ by $\sin\alpha$ where α is the half-angle of the cone.

(d) For ω to be equal to $\dot{\phi}_o$, it must be that $\sin\alpha = 1/\sqrt{3}$ or $\alpha = 35.3°$. If this is the case, the height z will return to its initial value in the time for one complete orbit around the cone. Therefore the orbit is closed and is (approximately at least) a circle, tilted at a small angle $\epsilon_{\max}/cz_o$.

13.18 ⋆⋆⋆ **(a)** According to (7.103), $\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 - q(V - \dot{\mathbf{r}}\cdot\mathbf{A})$. Therefore, the generalized momentum has $p_x = \partial\mathcal{L}/\partial\dot{x} = m\dot{x} + qA_x$, with similar expressions for p_y and p_z. Thus

$$\mathbf{p} = m\dot{\mathbf{r}} + q\mathbf{A} \quad \text{or} \quad \dot{\mathbf{r}} = \frac{\mathbf{p} - q\mathbf{A}}{m}. \tag{vii}$$

The Hamiltonian is

$$\mathcal{H} = \mathbf{p}\cdot\dot{\mathbf{r}} - \mathcal{L} = \mathbf{p}\cdot\frac{\mathbf{p}-q\mathbf{A}}{m} - \left[\frac{1}{2}m\left(\frac{\mathbf{p}-q\mathbf{A}}{m}\right)^2 - q\left(V - \frac{\mathbf{p}-q\mathbf{A}}{m}\cdot\mathbf{A}\right)\right]$$
$$= \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + qV$$

(b) Hamilton's equations are

$$\dot{x} = \frac{\partial\mathcal{H}}{\partial p_x} = \frac{p_x - qA_x}{m} \quad \text{and} \quad \dot{p}_x = -\frac{\partial\mathcal{H}}{\partial x} = q\left(\sum_{i=1}^{3}\dot{r}_i\frac{\partial A_i}{\partial x} - \frac{\partial V}{\partial x}\right)$$

with similar equations for y and z. Combining these two equations we find

$$m\ddot{x} = \dot{p}_x - q\frac{dA_x}{dt} = q\left(\sum_i \dot{r}_i\frac{\partial A_i}{\partial x} - \frac{\partial V}{\partial x}\right) - q\left(\sum_i\frac{\partial A_x}{\partial r_i}\dot{r}_i + \frac{\partial A_x}{\partial t}\right)$$
$$= q\left[-\left(\frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t}\right) + \dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]$$

which you can recognize as the x component of the Lorentz-force equation $m\ddot{\mathbf{r}} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$. Since the other two components work in exactly the same way, we're home.

13.20 $\star$ **(a)** $U(\mathbf{r}) = -\int \mathbf{F}\cdot d\mathbf{r} = -\mathbf{F}\cdot\mathbf{r}$. Therefore

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} - \mathbf{F}\cdot\mathbf{r} = \frac{p_x^2 + p_y^2}{2m} - F_x x - F_y y.$$

(b) If we choose our x axis in the direction of $\mathbf{F}$, then $F_y = 0$ and the coordinate y is ignorable.

(c) If neither axis is parallel to $\mathbf{F}$, then neither F_x nor F_y is zero, and neither $\partial\mathcal{H}/\partial x$ nor $\partial\mathcal{H}/\partial y$ is zero, so neither of the coordinates is ignorable.

13.23 $\star\star\star$ **(a)** The gravitational PE is $U_{\text{gr}} = Mgy - mgy - mg(x+y) + \text{const} = -mgx$ if we drop the uninteresting constant. The spring PE is harder. If we let l_o denote the spring's natural, unloaded length, then $k(l_\text{e} - l_\text{o}) = mg$ and if x' denotes the spring's true extension (from its unloaded length), then $l_\text{o} + x' = l_\text{e} + x$ so

$$x' = x + (l_\text{e} - l_\text{o}) = x + \frac{mg}{k}$$

Thus the spring PE is

$$U_{\text{spr}} = \tfrac{1}{2}kx'^2 = \tfrac{1}{2}k\left(x + \frac{mg}{k}\right)^2 = \tfrac{1}{2}kx^2 + mgx + \text{const.}$$

If we add this to $U_{\text{gr}} = -mgx$, the terms in mgx cancel and (dropping another uninteresting constant) we get $U = U_{\text{gr}} + U_{\text{spr}} = \frac{1}{2}kx^2$ as claimed.

(b) The KE is $T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m(\dot{x}+\dot{y})^2 = \frac{1}{2}m[3\dot{y}^2 + (\dot{x}+\dot{y})^2]$, from which we find the momenta,

$$p_x = \frac{\partial T}{\partial \dot{x}} = m(\dot{x}+\dot{y}) \qquad \text{and} \qquad p_y = \frac{\partial T}{\partial \dot{y}} = m(\dot{x}+4\dot{y})$$

whence

$$\dot{x} + \dot{y} = \frac{p_x}{m} \qquad \text{and} \qquad \dot{y} = \frac{1}{3m}(p_y - p_x).$$

From these we can calculate the Hamiltonian,

$$\mathcal{H} = T + U = \frac{1}{2m}\left[\frac{(p_x - p_y)^2}{3} + p_x^2\right] + \frac{1}{2}kx^2.$$

Because this doesn't depend on y, the coordinate y is ignorable. This is traceable to the fact that the total mass on each side is the same.

(c) The Hamilton equations for x are

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{1}{3m}(4p_x - p_y) \quad \text{and} \quad \dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = -kx \tag{ix}$$

and those for y

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{1}{3m}(p_y - p_x) \quad \text{and} \quad \dot{p}_y = -\frac{\partial \mathcal{H}}{\partial y} = 0. \tag{x}$$

The initial conditions are that $x(0) = x_o$, $y(0) = y_o$, and $\dot{x}(0) = \dot{y}(0) = 0$. These imply that $p_x(0) = p_y(0) = 0$, and, because p_y is constant, $p_y = 0$ for all time. Combining the two equations (ix) and setting $p_y = 0$, we find that $\ddot{x} = 4\dot{p}_x/3m = -4kx/3m$. Therefore $x = x_o \cos\omega t$, where $\omega = \sqrt{4k/3m}$. Next, from the first of Eqs.(ix) (with $p_y = 0$) we find that $p_x = \frac{3}{4}m\dot{x} = -\frac{3}{4}m\omega x_o \sin\omega t$ and finally, from the first of Eqs.(x), $\dot{y} = -p_x/3m = \frac{1}{4}\omega x_o \sin\omega t$, so $y = -\frac{1}{4}x_o \cos\omega t$ + const $= y_o + \frac{1}{4}x_o(1 - \cos\omega t)$.

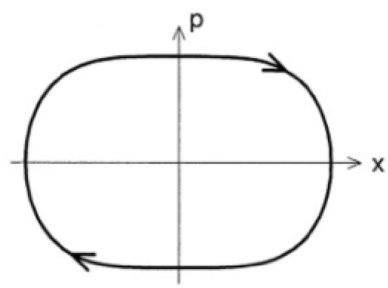
13.26 ⋆ The potential energy is

$$U = -\int F\,dx = \tfrac{1}{4}kx^4,$$

and the Hamiltonian is

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{4}kx^4 = E.$$

In the two-dimensional phase space, with coordinates x and p, this defines the flattened ellipse shown.



13.31 ⋆

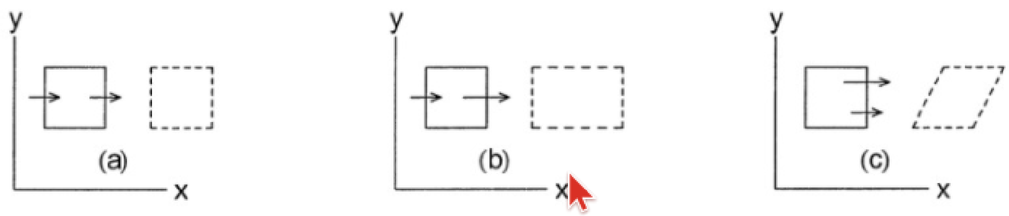
(a) If $\mathbf{v} = k\mathbf{r} = (kx, ky, kz)$, then $\nabla\cdot\mathbf{v} = k\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3k$

(b) If $\mathbf{v} = k(z, x, y)$, then $\nabla\cdot\mathbf{v} = k\left(\frac{\partial z}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial y}{\partial z}\right) = 0$

(c) If $\mathbf{v} = k(z, y, x)$, then $\nabla\cdot\mathbf{v} = k\left(\frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z}\right) = k$

(d) If $\mathbf{v} = k(x, y, -2z)$, then $\nabla\cdot\mathbf{v} = k\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} - 2\frac{\partial z}{\partial z}\right) = k(1 + 1 - 2) = 0$

13.32 ⋆⋆ **(a)** With $\mathbf{v} = (k, 0, 0)$, $\boldsymbol{\nabla}\cdot\mathbf{v} = \partial k/\partial x + 0 + 0 = 0$. This flow is sketched in the left picture. Fluid is flowing into the square on its left, but out on its right, and these two effects cancel. To put it another way, the square as a whole is moving to the right, but its front and back are moving at the same speed, so its volume doesn't change.



(b) With $\mathbf{v} = (kx, 0, 0)$, $\boldsymbol{\nabla}\cdot\mathbf{v} = k\partial x/\partial x = k$. This flow is shown in the middle picture. Fluid is flowing into the square on its left, and out on its right. Since the speed of flow is greater on the right, there is a net outflow. The square as a whole is stretching.

(c) With $\mathbf{v} = (ky, 0, 0)$, $\boldsymbol{\nabla}\cdot\mathbf{v} = k\partial y/\partial x = 0$. This flow is shown in the right picture. Fluid is flowing into the square on its left, and out on its right. The speed of flow is greater near the top, but the net flows on the left and right cancel. The square as a whole is becoming a parallelogram, but its volume isn't changing.

13.35 ⋆⋆ The initial volume occupied by the beam is $V_o = (\pi R_o^2 L_o)[\pi(\Delta p_\perp)^2 2\Delta p_z]$. (This is the volume in phase space.) By Liouville's theorem, this volume can't change. Thus when R shrinks (with L_o and Δp_z fixed), $\Delta p_\perp$ has to grow. In the long run, this means that R will increase again.
