11.2 ⋆⋆ Let x_1 and x_2 be the extensions of the two springs from their unstretched lengths, and x_{10} and x_{20} their values at equilibrium. The displacements from equilibrium are

$$y_1 = x_1 - x_{10} \quad \text{and} \quad y_2 = x_2 - x_{20}. \tag{i}$$

The net downward forces on the two masses are

$$F_1 = m_1 g - k_1 x_1 + k_2(x_2 - x_1) \quad \text{and} \quad F_2 = m_2 g - k_2(x_2 - x_1), \tag{ii}$$

and the conditions for equilibrium are

$$m_1 g = k_1 x_{10} - k_2(x_{20} - x_{10}) \quad \text{and} \quad m_2 g = k_2(x_{20} - x_{10}).$$

Using (i) to eliminate x_1 and x_2 from (ii), we find that

$$\begin{aligned} m_1\ddot{y}_1 = F_1 &= m_1 g - k_1(y_1 + x_{10}) + k_2[y_2 - y_1 + (x_{20} - x_{10})] \\ &= -k_1 y_1 + k_2(y_2 - y_1) \end{aligned}$$

where, in the second line I used the equilibrium condition to cancel several terms. Similarly

$$\begin{aligned} m_2\ddot{y}_2 = F_2 &= m_2 g - k_2(x_2 - x_1) \\ &= -k_2(y_2 - y_1). \end{aligned}$$

The last two results combine to give the matrix equation $\mathbf{M}\ddot{\mathbf{y}} = -\mathbf{K}\mathbf{y}$ where

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

11.4 ⋆⋆ **(a)** Putting $m_1 = m_2$ and $k_1 = k_3$ in (11.5), we find that the mass and spring-constant matrices are

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix}$$

and hence

$$(\mathbf{K} - \omega^2\mathbf{M}) = \begin{bmatrix} k_1 + k_2 - m\omega^2 & -k_2 \\ -k_2 & k_1 + k_2 - m\omega^2 \end{bmatrix}.$$

The determinant of the last matrix is $\det(\mathbf{K} - \omega^2\mathbf{M}) = (m\omega^2 - k_1)(m\omega^2 - k_1 - 2k_2)$. The two normal frequencies are the roots of the equation $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$ and are $\sqrt{k_1/m}$ and $\sqrt{(k_1 + 2k_2)/m}$. If I set $k_1 = k_2 = k$, these reduce to the results (11.15) for all three springs equal.

(b) The motion in each normal mode is determined by the vector $\mathbf{a}$ satisfying the eigenvector equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$. For $\omega = \omega_1$ this is easily seen to be exactly the same as for the equal-spring case; in particular, the motion is as given by (11.18) and as shown in Figure 11.2. The two carts oscillate in phase with equal amplitudes, so that the middle spring is undisturbed. This means its strength is irrelevant and we get the same motion with the same frequency whatever the value of k_2. For the second mode, with $\omega = \omega_2$, the motion is again the same as for the corresponding mode of the equal-spring case, namely (11.20) and Figure 11.4. This is a little subtle: In this mode, the middle spring does change length, and the frequency does depend on the value of k_2. Nevertheless, the motion is independent of k_2 since the symmetric arrangement, with the outside springs equally stretched and the middle one compressed, or vice versa, leads to equal (but opposite) forces on the two equal-mass carts and allows them to oscillate with equal amplitudes exactly out of phase.

11.6 ⋆⋆ **(a)** In this case

$$(\mathbf{K}-\omega^2\mathbf{M}) = \begin{bmatrix} 5k - m\omega^2 & -2k \\ -2k & 2k - m\omega^2 \end{bmatrix}$$

with determinant $\det(\mathbf{K}-\omega^2\mathbf{M}) = (m\omega^2 - k)(m\omega^2 - 6k)$. Thus the two normal frequencies are $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{6k/m}$.

(b) The motion in each normal mode is determined by the vector **a** satisfying the eigenvector equation $(\mathbf{K}-\omega^2\mathbf{M})\mathbf{a} = 0$. For $\omega = \omega_1$ this gives $a_2 = 2a_1$, so the two carts oscillate in phase, with the second cart's amplitude equal to twice that of the first. If $\omega = \omega_2$ then $a_2 = -a_1/2$, so the two carts oscillate exactly out of phase, with the second cart's amplitude equal to half that of the first.

11.9 ⋆⋆ **(a)** With identical masses and springs, the two equations of motion (11.2) are

$$m\ddot{x}_1 = -2kx_1 + kx_2 \qquad \text{and} \qquad m\ddot{x}_2 = kx_1 - 2kx_2\,.$$

If we add these two equations and define $\xi_1 = \frac{1}{2}(x_1 + x_2)$, we find that $m\ddot{\xi}_1 = -k\xi_1$. Similarly, if we subtract the second equation from the first and define $\xi_2 = \frac{1}{2}(x_1 - x_2)$, we get $m\ddot{\xi}_2 = -3k\xi_2$. These equations for ξ_1 and ξ_2 are uncoupled, as claimed.

(b) The general solutions of the equations of motion for ξ_1 and ξ_2 are $\xi_1 = A_1\cos(\omega_1 t - \delta_1)$ and $\xi_2 = A_2\cos(\omega_2 t - \delta_2)$, where A_1, A_2, δ_1 and δ_2 are arbitrary constants, $\omega_1 = \sqrt{k/m}$, and $\omega_2 = \sqrt{3k/m}$. Therefore

$$x_1 = \xi_1 + \xi_2 = A_1\cos(\omega_1 t - \delta_1) + A_2\cos(\omega_2 t - \delta_2)$$

and

$$x_2 = \xi_1 - \xi_2 = A_1\cos(\omega_1 t - \delta_1) - A_2\cos(\omega_2 t - \delta_2)$$

in agreement with Eq.(11.21).

11.12 ⋆⋆⋆ **(a)** The force of viscous drag is $\beta m(\dot{x}_2 - \dot{x}_1)$, to the right on cart 1 and to the left on cart 2. The equation of motion for cart 1 is

$$m\ddot{x}_1 = -kx_1 + \beta m(\dot{x}_2 - \dot{x}_1) \quad \text{or} \quad \ddot{x}_1 + \omega_o{}^2 x_1 + \beta\dot{x}_1 - \beta\dot{x}_2 = 0$$

and that for cart 2 is

$$m\ddot{x}_2 = -kx_2 - \beta m(\dot{x}_2 - \dot{x}_1) \quad \text{or} \quad \ddot{x}_2 + \omega_o{}^2 x_2 - \beta\dot{x}_1 + \beta\dot{x}_2 = 0.$$

These two equations combine as a single matrix equation $\ddot{\mathbf{x}} + \omega_o{}^2\mathbf{x} + \beta\mathbf{D}\dot{\mathbf{x}} = 0$, where **D** and **x** are the matrices

$$\mathbf{D} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) If we substitute the proposed complex solution $\mathbf{z}(t) = \mathbf{a}e^{rt}$ into the matrix equation of motion, we find the **a** must satisfy

$$[(r^2 + \omega_o^2)\mathbf{1} + \beta r\mathbf{D}]\mathbf{a} = 0. \tag{viii}$$

This has nontrivial solutions only if the determinant of the matrix in brackets is zero. That is,

$$\det[(r^2 + \omega_o^2)\mathbf{1} + \beta r\mathbf{D}] = (r^2 + \omega_o^2)(r^2 + \omega_o^2 + 2\beta r) = 0$$

Thus the values of r that give a solution are $r = r_1 = i\omega_o$ and $r = r_2 = -\beta + i\sqrt{\omega_o^2 - \beta^2} = -\beta + i\omega_1$. [There are actually two more solutions with the opposite sign to the imaginary part, but these give the same actual motion $\mathbf{x}(t)$.] If we put $r = r_1$ in (viii) we find that $a_1 = a_2 = Ae^{-i\delta}$, say. Thus the first mode has $x_1(t) = x_2(t) = A\cos(\omega_o t - \delta)$, and the two carts move together with equal amplitudes. Because cart 2 is stationary with respect to cart 1, the drag force is zero and the motion is undamped. If we put $r = r_2$ in (viii) we find that $a_1 = -a_2 = Ae^{-i\delta}$, say, and the second mode has $x_1(t) = -x_2(t) = A\cos(\omega_1 t - \delta)e^{-\beta t}$. In this mode the two carts move in opposite directions and the drag force causes the motion to damp out.

11.14 ⋆⋆ **(a)** The kinetic energy is $T = \frac{1}{2}mL^2(\dot\phi_1^2 + \dot\phi_2^2)$. The gravitational potential energy of either pendulum has the form $mgL(1 - \cos\phi) \approx \frac{1}{2}mgL\phi^2$, and the spring's PE is $\frac{1}{2}kx^2 \approx \frac{1}{2}kL^2(\phi_2 - \phi_1)^2$. Putting these together,

$$\mathcal{L} = \tfrac{1}{2}mL^2(\dot\phi_1^2 + \dot\phi_2^2) - \tfrac{1}{2}mgL(\phi_1^2 + \phi_2^2) - \tfrac{1}{2}kL^2(\phi_2 - \phi_1)^2$$

from which we get the Lagrange equations:

$$\ddot\phi_1 = -\omega_o{}^2\phi_1 + (k/m)(\phi_2 - \phi_1)$$
$$\ddot\phi_2 = -\omega_o{}^2\phi_2 - (k/m)(\phi_2 - \phi_1)$$

where I have divided through by mL^2 and introduced the natural frequency for either pendulum (without the spring) given by $\omega_o{}^2 = g/L$.

(b) From the equations of motion, you can write down the "mass matrix" **M** and "spring matrix" **K**, and thence the matrix

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} \omega_o{}^2 + k/m - \omega^2 & -k/m \\ -k/m & \omega_o{}^2 + k/m - \omega^2 \end{bmatrix}.$$

The determinant of this matrix is $(\omega_o{}^2 - \omega^2)(\omega_o{}^2 + 2k/m - \omega^2)$, and the two normal frequencies are

$$\omega_1 = \omega_o \qquad \text{and} \qquad \omega_2 = \sqrt{\omega_o{}^2 + 2k/m}.$$

The corresponding motions are found by solving the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ with ω set equal to ω_1 and ω_2 in turn. For the first mode, this gives the eigenvector $\mathbf{a} = A(1, 1)$ (actually a 2×1 column, of course). This means that in the first mode the two pendulums oscillate in unison (in phase with equal amplitudes). In this mode the spring is unstretched, its presence is irrelevant, and the frequency is just the natural frequency for a single pendulum.

For the second mode, $\mathbf{a} = A(1, -1)$, and the two pendulums oscillate with equal amplitudes but exactly out of phase. Notice that, in either mode, the two pendulums behave just like the two carts of Section 11.2.

11.16 ⋆⋆ **(a)** The matrices $\mathbf{M}$ and $\mathbf{K}$ are given in Eq.(11.44) and the determinant of $\mathbf{K}-\omega^2\mathbf{M}$ is easily evaluated to give

$$\det(\mathbf{K}-\omega^2\mathbf{M}) = m_2L_1L_2[m_1L_1L_2\omega^4 - Mg(L_1+L_2)\omega^2 + Mg^2]$$

where $M = m_1+m_2$. The normal frequencies are found by setting this equal to zero and are

$$\omega^2 = \frac{Mg(L_1+L_2) \pm \sqrt{M^2g^2(L_1+L_2)^2 - 4m_1ML_1L_2g^2}}{2m_1L_1L_2} \tag{x}$$

(b) Putting $m_1 = m_2$ and $L_1 = L_2 = L$, we find $\omega^2 = (2\pm\sqrt{2})g/L$ in agreement with (11.47).

(c) In the limit that $m_2 \to 0$, Eq.(x) becomes

$$\omega^2 = \frac{g}{2L_1L_2}[(L_1+L_2)\pm(L_1-L_2)] = \frac{g}{L_2} \text{ or } \frac{g}{L_1}.$$

The first of these corresponds to the very light lower pendulum oscillating at its natural frequency while the upper remains unaffected and stationary. The second has the upper heavy pendulum swinging at its natural frequency, unaffected by the presence of the very light lower one.

11.22 ⋆ The position of any one mass relative to its equilibrium position is

$$(x,y) = [L\sin\phi, L(1-\cos\phi)]$$

The gravitational PE of mass 1 is therefore $U_1^{\text{gr}} = mgL(1-\cos\phi_1)$, with corresponding expressions for the other two masses. The length of the first spring is

$$d_1 = \sqrt{(d_o + L\sin\phi_2 - L\sin\phi_1)^2 + (L\cos\phi_2 - L\cos\phi_1)^2} \tag{xiv}$$

where d_o is the unstretched length of either spring. The PE of spring 1 is therefore

$$U_1^{\text{sp}} = \tfrac{1}{2}k(d_1-d_o)^2 = \tfrac{1}{2}k\left(\sqrt{(d_o + L\sin\phi_2 - L\sin\phi_1)^2 + (L\cos\phi_2 - L\cos\phi_1)^2} - d_o\right)^2 \tag{xv}$$

with a corresponding expression for U_2^{sp} (just replace ϕ_2 and ϕ_1 by ϕ_3 and ϕ_2). The total PE is then

$$U = U_1^{\text{gr}} + U_2^{\text{gr}} + U_3^{\text{gr}} + U_1^{\text{sp}} + U_2^{\text{sp}}. \tag{xvi}$$

If all three angles are small the gravitational PE's simplify as usual to $U_1^{\text{gr}} \approx \frac{1}{2}mgL\phi_1^2$ and so on. The length (xiv) becomes $d_1 \approx d_o + L\phi_2 - L\phi_1$ and the PE (xv) becomes $U_1^{\text{sp}} \approx \frac{1}{2}kL^2(\phi_2-\phi_1)^2$ with a corresponding expression for U_2^{sp}. Putting all of these into (xvi) (and setting $m = L = 1$) we obtain the approximate expression (11.68).

11.24 ⋆⋆ If we let the strings have length L_o when both masses are in their equilibrium positions, then when the first mass is at position y_1 the length of the first string is $L = \sqrt{L_o^2 + y_1^2} \approx L_o(1 + \frac{1}{2}y_1^2/L_o^2)$. The string is stretched by an amount $d = \frac{1}{2}y_1^2/L_o$ and, with the tension T essentially constant, its PE is therefore $Td = \frac{1}{2}y_1^2 T/L_o$. Using the same argument for the other two strings, you can check that the total PE is

$$U = \tfrac{1}{2}[y_1^2 + (y_1 - y_2)^2 + y_2^2]T/L_o = [y_1^2 - y_1y_2 + y_2^2]T/L_o$$

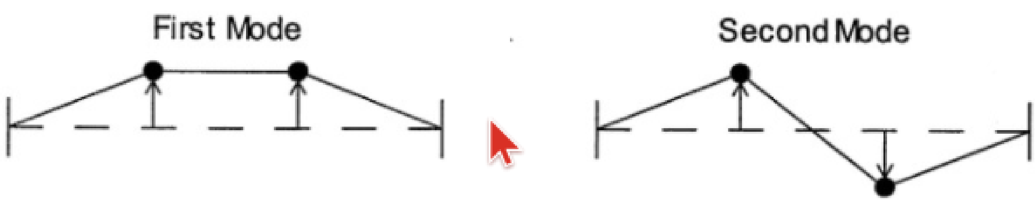
and the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}m(\dot{y}_1^2 + \dot{y}_2^2) - [y_1^2 - y_1y_2 + y_2^2]T/L_o.$$

The matrix $(\mathbf{K} - \omega^2\mathbf{M})$ is

$$(\mathbf{K} - \omega^2\mathbf{M}) = \begin{bmatrix} 2T/L_o - \omega^2 m & -T/L_o \\ -T/L_o & 2T/L_o - \omega^2 m \end{bmatrix}.$$

The eigenvalues are $\omega_1^2 = T/(mL_o)$ and $\omega_2^2 = 3T/(mL_o)$. For the first mode, the vector that describes the motion is $\mathbf{a} = A(1,1)$ (actually a 2×1 column), and the two masses oscillate in unison. For the second mode $\mathbf{a} = A(1,-1)$, and the two masses oscillate with equal amplitudes but exactly out of phase as shown on the next page.



11.25 ⋆⋆ The mass and spring constant matrices are

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}$$

from which we find

$$\det(K - \omega^2 M) = m^3(2\omega_o^2 - \omega^2)\left[(2-\sqrt{2})\omega_o^2 - \omega^2\right]\left[(2+\sqrt{2})\omega_o^2 - \omega^2\right]$$

where I have introduced the shorthand, $\omega_o = \sqrt{k/m}$. Thus the normal frequencies are

$$\omega_1 = \omega_o\sqrt{2-\sqrt{2}}, \quad \omega_2 = \omega_o\sqrt{2}, \quad \text{and} \quad \omega_3 = \omega_o\sqrt{2+\sqrt{2}}$$

In the first mode, the eigenvector $\mathbf{a}$ has $a_1 = a_3 = a_2/\sqrt{2}$, so all three carts oscillate in phase, with the middle cart's amplitude $\sqrt{2}$ bigger than the outer two. In the second, $a_2 = 0$, while $a_1 = -a_3$, so the middle cart is stationary, while the first and third oscillate exactly out of phase. In the third mode, $a_1 = a_3 = -a_2/\sqrt{2}$, so the first and third carts oscillate in phase, while the middle one is exactly out of phase with amplitude $\sqrt{2}$ times bigger.

11.28 ⋆⋆ **(a)** For small oscillations, the KE is

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}M(\dot{x}+L\dot{\phi})^2 = \tfrac{1}{2}(m+M)\dot{x}^2 + ML\dot{x}\dot{\phi} + \tfrac{1}{2}ML^2\dot{\phi}^2$$

while the PE is $U = MgL(1-\cos\phi) \approx \frac{1}{2}MgL\phi^2$. Thus if we take x and ϕ as our coordinates (in that order) the matrices **M** and **K** are

$$\mathbf{M} = \begin{bmatrix} m+M & ML \\ ML & ML^2 \end{bmatrix} = M\begin{bmatrix} 1+\lambda & L \\ L & L^2 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 0 & 0 \\ 0 & MgL \end{bmatrix} = M\begin{bmatrix} 0 & 0 \\ 0 & L^2{\omega_o}^2 \end{bmatrix}$$

where I have introduced the mass ratio $\lambda = m/M$ and the frequency $\omega_o = \sqrt{g/L}$. Therefore

$$(\mathbf{K}-\omega^2\mathbf{M}) = -M\begin{bmatrix} \omega^2(1+\lambda) & \omega^2 L \\ \omega^2 L & (\omega^2-{\omega_o}^2)L^2 \end{bmatrix}$$

and (as you can check) $\det(\mathbf{K}-\omega^2\mathbf{M}) = ML^2\omega^2[\lambda\omega^2-(1+\lambda)\omega_o^2]$. The normal frequencies are $\omega_1 = 0$ and $\omega_2 = \omega_o\sqrt{(1+\lambda)/\lambda}$.

(b) If we set $\omega = \omega_1 = 0$, the equation $(\mathbf{K}-\omega^2\mathbf{M})\mathbf{a} = 0$ reduces to $\mathbf{Ka} = 0$, which requires that $a_2 = 0$; that is, $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (times any constant). If we try a solution of the form $\mathbf{x}(t) = \mathbf{a}f(t)$, then the equation of motion $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{Kx}$ becomes $\ddot{f} = 0$, so that $f(t) = x_o + v_o t$. In this mode, the cart moves with constant velocity, while the pendulum is stationary relative to the cart and hanging vertically.

If we set $\omega = \omega_2$, the equation $(\mathbf{K}-\omega^2\mathbf{M})\mathbf{a} = 0$ requires that $(1+\lambda)a_1 = -La_2$, so $x = A\cos(\omega_2 t-\delta)$ and $\phi = -A\cos(\omega_2 t-\delta)(1+\lambda)/L$. In this mode, the cart and bob oscillate in opposite directions leaving their CM stationary.

11.32 ⋆⋆⋆ **(a)** I'll introduce the notation $\lambda = M/m$ for the ratio of the two different masses, and I'll use units with $m = k = 1$. With this arrangement, any frequencies are measured in units of $\omega_o = \sqrt{k/m}$, the natural frequency of a mass m on a spring k. The total KE is $T = \frac{1}{2}(\dot{x}_1^2 + \lambda\dot{x}_2^2 + \dot{x}_3^2)$ and the PE is $U = \frac{1}{2}k(x_1 - x_2)^2 + \frac{1}{2}k(x_2 - x_3)^2$. The matrices $\mathbf{M}$ and $\mathbf{K}$ are

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} 1-\omega^2 & -1 & 0 \\ -1 & 2-\lambda\omega^2 & -1 \\ 0 & -1 & 1-\omega^2 \end{bmatrix}.$$

Therefore, as you can check, $\det(\mathbf{K} - \omega^2\mathbf{M}) = -\omega^2(\omega^2 - 1)(\lambda\omega^2 - 2 - \lambda)$ and the normal frequencies are $\omega_1 = 0$, $\omega_2 = 1$, and $\omega_3 = \sqrt{(2+\lambda)/\lambda}$. (In arbitrary units the last two would be $\omega_2 = \omega_o$, and $\omega_3 = \sqrt{(2+\lambda)/\lambda}\,\omega_o$.)

(b) If we put $\omega = \omega_2 = 1$ in the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$, we find that $a_2 = 0$ and $a_1 = -a_3$. Thus, in the second mode the center atom is stationary, while the outer two oscillate with frequency ω_o and equal amplitudes but 180° out of phase. If we put $\omega = \omega_3$ in the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$, we find that $a_1 = a_3 = -a_2(\lambda/2)$. Thus, in the third mode, atoms 1 and 3 oscillate in phase with the same amplitude, while the center atom oscillates 180° out of phase with amplitude $2/\lambda$ times that of the others.

(c) If we put $\omega = 0$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ becomes $\mathbf{Ka} = 0$, and we find that $a_1 = a_2 = a_3$. As in Problem 11.27, if we try a solution of the form $\mathbf{x} = \mathbf{a}f(t)$, the equation of motion implies that $\ddot{f} = 0$, so all three atoms move with the same constant velocity separated by their equilibrium separation.

11.34 ⋆⋆

$$\mathbf{a}_{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{a}_{(3)} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}. \tag{xxi}$$

If $\mathbf{x}$ is an arbitrary 3×1 column (with elements x_1, x_2, x_3, or ϕ_1, ϕ_2, ϕ_3 for pendulums) then we can try expanding $\mathbf{x}$ in terms of the three eigenvectors (xxi),

$$\mathbf{x} = \xi_1 \mathbf{a}_{(1)} + \xi_2 \mathbf{a}_{(2)} + \xi_3 \mathbf{a}_{(3)}. \tag{xxii}$$

This gives three equations for the three coefficients ξ_1, ξ_2, ξ_3,

$$\xi_1 + \xi_2 + \xi_3 = x_1, \quad \xi_1 - 2\xi_3 = x_2 \quad \text{and} \quad \xi_1 - \xi_2 + \xi_3 = x_3.$$

These are easily solved to give

$$\xi_1 = \frac{x_1 + x_2 + x_3}{3}, \quad \xi_2 = \frac{x_1 - x_3}{2}, \quad \text{and} \quad \xi_3 = \frac{x_1 - 2x_2 + x_3}{6}.$$

If you substitute these values into the right side of Eq.(xxii), you can check that you do indeed get $\mathbf{x}$. Thus we have successfully expanded an arbitrary $\mathbf{x}$ in terms of the eigenvectors (xxi).

Normal coordinates are supposed to have the property that any one of them can oscillate while the other two remain zero and that this motion is one of the normal modes. For example, if ξ_1 oscillates while ξ_2 and ξ_3 remain zero, then the condition $\xi_2 = 0$ implies that $x_1 = x_3$ and then the condition $\xi_3 = 0$ implies that $x_1 = x_2 = x_3$. When $\xi_1 = (x_1 + x_2 + x_3)/3$ oscillates, this means that x_1, x_2, and x_3 all oscillate in phase with equal amplitudes, and this is indeed the first mode of Fig.11.14. You can check that the other two modes work similarly.