10.2 ⋆ Let us denote by T_1 the KE of the motion of the CM plus the rotational KE about the CM:

$$T_1 = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I_{\rm cm}\omega^2 = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}(\tfrac{1}{2}MR^2)(v/R)^2 = \tfrac{3}{4}Mv^2$$

where, for the second equality, I used $I_{\rm cm} = \frac{1}{2}MR^2$ and $\omega = v/R$. On the other hand, the rotational KE about the instantaneous point of contact P is (Note that ω is the same either way.)

$$T_2 = \tfrac{1}{2}I_P\omega^2 = \tfrac{1}{2}(\tfrac{3}{2}MR^2)(v/R)^2 = \tfrac{3}{4}Mv^2.$$

These two are clearly equal.

10.6 ⋆⋆ **(a)** I'll choose my origin at the center of the hemisphere with the hemisphere in the region $z > 0$. For the same reasons as in Problem 10.5, $X = Y = 0$, and

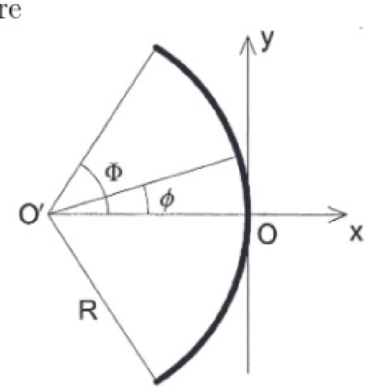
$$Z = \frac{1}{M}\int \varrho z dV = \frac{\varrho}{M}\int_a^b r^2 dr \int_0^{\pi/2} \sin\theta d\theta \int_0^{2\pi} d\phi\, r\cos\theta$$
$$= \frac{1}{V}\int_a^b r^3 dr \int_0^{\pi/2} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi \;=\; \frac{3(b^4 - a^4)}{8(b^3 - a^3)}$$

since $V = \frac{2}{3}\pi(b^3 - a^3)$.

(b) If $a = 0$, the answer reduces to $Z = 3b/8$, which is the correct CM position for a solid hemisphere of radius b. (See Problem 10.5.)

(c) If $b \to a$, we can write $b = a + \epsilon$ and the binomial expansion reduces the answer to $Z = (3 \times 4a^3\epsilon)/(8 \times 3a^2\epsilon) = \frac{1}{2}a$. This is the CM position of a thin hemispherical shell of radius a.

10.8 ⋆⋆ I'll use a temporary origin O' at the center of curvature of the bent wire, as shown. Let's consider a short segment of wire at angle ϕ, where ϕ runs from $-\Phi$ to Φ and $2R\Phi = L$. If the segment subtends an angle $d\phi$ at O', its mass is $dm = Md\phi/2\Phi = MRd\phi/L$ and its x' coordinate (relative to the origin O') is $x' = R\cos\phi$. By symmetry, the CM has $Y = Z = 0$, and



$$X' = \frac{1}{M}\int x' dm = \frac{R^2}{L}\int_{-\Phi}^{\Phi} \cos\phi d\phi$$
$$= \frac{2R^2}{L}\sin\Phi = \frac{2R^2}{L}\sin(L/2R).$$

Thus the CM position relative to the origin O is $X = X' - R = R\left[\frac{2R}{L}\sin\left(\frac{L}{2R}\right) - 1\right]$.

If $R \to \infty$, the wire returns to its straight configuration and $L/2R \to 0$. Now, when t is small, $(1/t)\sin t \approx 1 - t^2/6$, so $X \approx R(L/2R)^2/6 \to 0$. This correctly reflects that the CM returns to the origin O.

If $2\pi R = L$, $L/2R = \pi$ and our answer reduces to $X = -R$. This is correct because the wire is now a single complete circle and the CM is at its center at $X = -R$.

10.12 ⋆⋆ The area of the triangular ends is $A = \sqrt{3}a^2$ and

$$I_{zz} = \frac{M}{V}\int (x^2+y^2)dV = \frac{M}{A}\int x^2dx\int dy + \frac{M}{A}\int dx\int y^2dy. \tag{i}$$

These two integrals take a little care (You need to draw a picture and decide on the limits of integration.) The result is that they are equal and

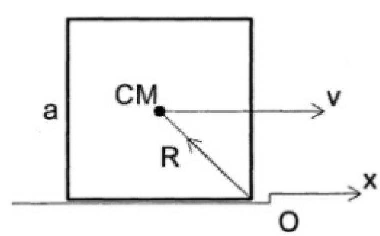
$$\int x^2dx\int dy = \int dx\int y^2dy = \sqrt{3}a^4/6.$$

Substituting into Eq.(i), we get $I_{zz} = \frac{1}{3}Ma^2$. The two products are zero, $I_{xz} = I_{yz} = 0$, because the prism has reflection symmetry in the xy plane.

10.14 ⋆⋆ **(a)** The moment of inertia of the wheel is $I_\text{w} = \frac{1}{2}M_\text{w}R_\text{w}^2 = 0.05$ kg·m^2, and, from Problem 10.11, that of the ship is $I_\text{s} = 2M_\text{s}(b^5-a^5)/5(b^3-a^3) = 1.23\times 10^5$ kg·m^2. By conservation of angular momentum, $\omega_\text{s} = \omega_\text{w}I_\text{w}/I_\text{s} = 4.08\times 10^{-4}$ rpm, and the time to turn through $\Delta\theta = 10°$ is $\Delta\theta/\omega_\text{s} = 68$ minutes.

(b) The KE given to the flywheel is $T_\text{w} = \frac{1}{2}I_\text{w}\omega_\text{w}^2 = 274$ J, and, as you can easily check, that of the ship is totally negligible. If the designers of the ship were very conscientious, they could have arranged that this energy could be recouped when the turn is complete (by using it to recharge a battery, for example), but, assuming it goes to waste, the energy needed for the whole maneuver is 274 J.

10.16 ⋆⋆ **(a)** The moment of inertia of the cube about any edge is worked out in Example 10.2 and is given by (10.47) as $\frac{2}{3}Ma^2$. During the collision, kinetic energy will be lost (the collision is inevitably inelastic), but the angular momentum L_y about the edge of the step is conserved. (I take the x direction as that of the incident velocity, as shown, and y into the page.) Just before the collision the angular momentum is $\sum m_\alpha \mathbf{r}_\alpha \times \mathbf{v} = M\mathbf{R}\times\mathbf{v}$, so that $L_y = Mav/2$. Just after the collision, the cube is rotating about the edge of the step and $L_y = I_{yy}\omega_\text{o} = \frac{2}{3}Ma^2\omega_\text{o}$. Equating these two expressions for L_y, we find that $\omega_\text{o} = 3v/(4a)$.



(b) If the initial speed is small, the cube's rotational motion about O will stop before the CM has passed the step, and the cube will fall backward. If v is big enough, the CM will pass the step and the cube will roll forward. At the critical speed that divides these possibilities, the CM will just come to rest vertically above O. Since mechanical energy is conserved in the rotational phase of motion, this critical speed is determined by the condition

$$\tfrac{1}{2}I_{yy}{\omega_\text{o}}^2 + Mga/2 = Mga/\sqrt{2}.$$

(The height of the CM above O is $a/2$ initially, and $a/\sqrt{2}$ when the CM is vertically above O.) Substituting for ω_o from part (a), we can solve for v and find $v_\text{crit} = [8(\sqrt{2}-1)ga/3]^{1/2}$.

10.18 ⋆⋆⋆ **(a)** The angular momentum (about the pivot) just after impact is $L = \Gamma dt = F b dt = \xi b$. This is the same as $I\omega$, so $\omega = \xi b/I$. Therefore the CM velocity is $v_{\rm cm} = a\omega = a\xi b/I$, and the total momentum is $P = mv_{\rm cm} = ma\xi b/I$.

(b) The total impulse delivered to the rod is $\xi + \eta$ and this is equal to the total momentum $P = ma\xi b/I$. Therefore, $\eta = P - \xi = (mab/I - 1)\xi$.

(c) The impulse at the pivot is zero if and only if $mab/I = 1$, so the sweet spot is at $b_{\rm o} = I/ma$.

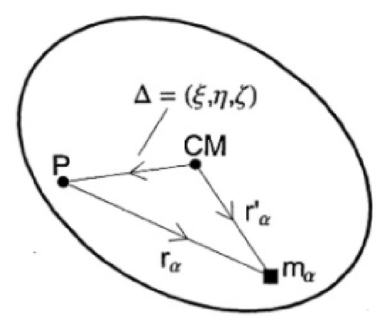
10.20 ⋆ If we use A to denote the set of all points of the first body, then A is the union $A = B \cup C$ and, from the definition (10.37),

$$I^A_{xx} = \sum_{\alpha\in A} m_\alpha(y_\alpha^2 + z_\alpha^2) = \sum_{\alpha\in B} m_\alpha(y_\alpha^2 + z_\alpha^2) + \sum_{\alpha\in C} m_\alpha(y_\alpha^2 + z_\alpha^2) = I^B_{xx} + I^C_{xx}.$$

The other two diagonal elements of $\mathbf{I}$ work in the same way, as do the six off-diagonal elements as defined by Eq.(10.38). Therefore, $\mathbf{I}^A = \mathbf{I}^B + \mathbf{I}^C$, as claimed.

If A is the result of removing C from B, then $B = A \cup C$ and, as above, $\mathbf{I}^B = \mathbf{I}^A + \mathbf{I}^C$, whence $\mathbf{I}^A = \mathbf{I}^B - \mathbf{I}^C$.

10.24 ⋆⋆ **(a)** For rotation about P, the moment of inertia $I_{xx} = \sum m_\alpha(y_\alpha^2 + z_\alpha^2)$. From the picture, you can see that $\mathbf{r}_\alpha = \mathbf{r}'_\alpha - \boldsymbol{\Delta}$, so that $x_\alpha = x'_\alpha - \xi$ and so on. Therefore



$$\begin{aligned} I_{xx} &= \sum m_\alpha[(y'_\alpha - \eta)^2 + (z'_\alpha - \zeta)^2] \\ &= \sum m_\alpha(y'^2_\alpha + z'^2_\alpha) + \sum m_\alpha(\eta^2 + \zeta^2) \\ &\quad - 2\eta\sum m_\alpha y'_\alpha - 2\zeta\sum m_\alpha z'_\alpha. \end{aligned}$$

The first sum on the second line is just $I^{\rm cm}_{xx}$. The second is $M(\eta^2 + \zeta^2)$, and the last two are zero by (10.7). Thus

$$I_{xx} = I^{\rm cm}_{xx} + M(\eta^2 + \zeta^2) \tag{iv}$$

as claimed. The other two diagonal elements work the same way, as do the six off-diagonal terms; for instance,

$$I_{yz} = I^{\rm cm}_{yz} - M\eta\zeta. \tag{v}$$

(b) In Example 10.2(b) we found $\mathbf{I}^{\rm cm}$ for a cube in (10.54), which gives

$$I^{\rm cm}_{xx} = \tfrac{1}{6}Ma^2 \qquad \text{and} \qquad I^{\rm cm}_{yz} = 0.$$

In part (a) of the same example, we found $\mathbf{I}$ for the same cube rotating about a corner, which is displaced from the CM by $\boldsymbol{\Delta} = (-a/2, -a/2, -a/2)$. There we found in (10.49)

$$I_{xx} = \tfrac{2}{3}Ma^2 = \tfrac{1}{6}Ma^2 + 2M(-a/2)^2 \qquad \text{and} \qquad I_{yz} = -\tfrac{1}{4}Ma^2 = 0 - M(-a/2)(-a/2).$$

As you can easily see these are precisely the relations (iv) and (v) with $\eta = \zeta = -a/2$.

10.30 $\star$ Choose the xy plane to contain the lamina. Then, according to Problem 10.23,

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & 0 \\ I_{yx} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}.$$

If $\boldsymbol{\omega}$ points along the z axis, then $\boldsymbol{\omega}$ has components $(0, 0, \omega)$, and $\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = (0, 0, I_{zz}\omega)$ (both to be thought of as column vectors), which also points along the z axis. Therefore the z axis is a principal axis.

10.32 $\star\star$ **(a)** Let us choose the principal axes as our coordinate directions. Then

$$\lambda_1 = \int \varrho(y^2 + z^2)dV \quad \text{and} \quad \lambda_2 = \int \varrho(x^2 + z^2)dV.$$

Adding these two equations, we get

$$\lambda_1 + \lambda_2 = \int \varrho(x^2 + y^2)dV + 2\int \varrho z^2 dV \geq \int \varrho(x^2 + y^2)dV = \lambda_3.$$

(b) The "$\geq$" in the above relations is an "$=$" if and only if all parts of the body have $z = 0$, that is, the body is a lamina lying in the plane $z = 0$.

10.36 $\star\star$ **(a)** The three masses are equal, $m_1 = m_2 = m_3 = m$ and their positions are

$$\mathbf{r}_1 = a(1, 0, 0), \quad \mathbf{r}_2 = a(0, 1, 2), \quad \text{and} \quad \mathbf{r}_3 = a(0, 2, 1).$$

Therefore

$$\left.\begin{aligned} I_{xx} &= \sum m_\alpha(y_\alpha^2 + z_\alpha^2) = ma^2(0 + 5 + 5) = 10ma^2 \\ I_{yy} &= \sum m_\alpha(x_\alpha^2 + z_\alpha^2) = ma^2(1 + 4 + 1) = 6ma^2 \\ I_{zz} &= \sum m_\alpha(x_\alpha^2 + y_\alpha^2) = ma^2(1 + 1 + 4) = 6ma^2 \\ I_{xy} &= -\sum m_\alpha x_\alpha y_\alpha = -ma^2(0 + 0 + 0) = 0 \\ I_{xz} &= -\sum m_\alpha x_\alpha z_\alpha = -ma^2(0 + 0 + 0) = 0 \\ I_{yz} &= -\sum m_\alpha y_\alpha z_\alpha = -ma^2(0 + 2 + 2) = -4ma^2 \end{aligned}\right\} \quad \text{or} \quad \mathbf{I} = 2ma^2 \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

(b) As you can check, the characteristic equation is

$$\det(\mathbf{I} - \lambda\mathbf{1}) = (10ma^2 - \lambda)^2(2ma^2 - \lambda) = 0$$

Therefore, the principal moments are $\lambda_1 = \lambda_2 = 10ma^2$ and $\lambda_3 = 2ma^2$. If we set $\lambda = 10ma^2$, the equation $(\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega} = 0$ yields three equations, $0 = 0$, $\omega_2 + \omega_3 = 0$, and $\omega_2 + \omega_3 = 0$, of which only one is independent. Thus there are two independent eigenvectors with $\lambda = 10ma^2$, which we can take to be $\mathbf{e}_1 = (1, 0, 0)$ and $\mathbf{e}_2 = (0, 1, -1)/\sqrt{2}$ or any other two perpendicular directions in the plane of these two. If we set $\lambda = 2ma^2$, the equation $(\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega} = 0$ yields three equations, $\omega_1 = 0$, $\omega_2 - \omega_3 = 0$, and $-\omega_2 + \omega_3 = 0$. There is just one independent eigenvector with $\lambda = 2ma^2$, which we can take to be $\mathbf{e}_3 = (0, 1, 1)/\sqrt{2}$.

10.40 ⋆⋆ **(a)** Multiplying the first of Equations (10.88) by $\lambda_1\omega_1$, the left side becomes $\lambda_1^2\omega_1\dot{\omega}_1$, which is the same as $\frac{1}{2}d(\lambda_1^2\omega_1^2)/dt$. Therefore

$$\frac{d}{dt}\left(\lambda_1^2\omega_1^2\right) = 2\lambda_1(\lambda_2 - \lambda_3)\omega_1\omega_2\omega_3.$$

Similarly, the second and third equations give

$$\frac{d}{dt}\left(\lambda_2^2\omega_2^2\right) = 2\lambda_2(\lambda_3 - \lambda_1)\omega_1\omega_2\omega_3 \quad \text{and} \quad \frac{d}{dt}\left(\lambda_3^2\omega_3^2\right) = 2\lambda_3(\lambda_1 - \lambda_2)\omega_1\omega_2\omega_3.$$

Adding these three equations and remembering that $\mathbf{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$, we find that $d\mathbf{L}^2/dt = 0$.

(b) If, instead, we multiply the first of Equations (10.88) by ω_1, we find that

$$\frac{1}{2}\frac{d}{dt}\left(\lambda_1\omega_1^2\right) = (\lambda_2 - \lambda_3)\omega_1\omega_2\omega_3.$$

Adding this to the corresponding two equations for the second and third components, we find that

$$\frac{1}{2}\frac{d}{dt}\left(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2\right) = \frac{d}{dt}T_{\text{rot}} = 0.$$

10.42 ⋆ The inertia tensor for the book (sides $a = 30$, $b = 20$, and $c = 3$, all in cm) can be evaluated as in Example 10.2. With the origin at the CM, all off-diagonal elements are zero, and the diagonal elements (which are the principal moments) are $\lambda_1 = M(b^2 + c^2)/12$, and so on. If the book's spin axis is close to the shortest symmetry axis (the z axis), then according to (10.91) the frequency of wobble is given by

$$\Omega^2 = \frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1\lambda_2}\omega_3^2 = \frac{(a^2 + b^2 - b^2 - c^2)(a^2 + b^2 - c^2 - a^2)}{(b^2 + c^2)(c^2 + a^2)}\omega_3^2$$
$$= \frac{(a^2 - c^2)(b^2 - c^2)}{(a^2 + c^2)(b^2 + c^2)}\omega_3^2 \qquad \text{(xvii)}$$

Putting in the given numbers, we find $\Omega = 0.968\omega_3 = 174$ rpm. If the book is spinning about the longest (x) axis we have only to swap λ_1 and λ_3, and we find $\Omega = 0.614\omega_3 = 111$ rpm.

10.45 ⋆⋆ **(a)** From Equation (10.93) the rate of precession of $\boldsymbol{\omega}$ about the earth's axis $\mathbf{e}_3$ is $\Omega_{\rm b} = \omega_3(\lambda_1 - \lambda_3)/\lambda_1 = 0.00327\omega_3$. The period of this precession is

$$\tau_{\rm b} = \frac{2\pi}{\Omega_{\rm b}} = \frac{1}{0.00327}\frac{2\pi}{\omega_3} = 306 \text{ days}$$

because $2\pi/\omega_3 = 1$ day. This is very nearly, but not quite, the claimed 305 days. The discrepancy is because $2\pi/\omega_3$ is actually 1 sidereal day (the time for one rotation of the earth relative to the stars), and a sidereal day is less than a solar day (what we normally consider to be a day) by about one part in 365. Thus 306 sidereal days are equal to about 305 solar days.

(b) From Fig.10.9 we see that $\tan\alpha = \omega_{\rm o}/\omega_3$. Since α is tiny ($\alpha = 0.2$ arcseconds $\approx 10^{-6}$ rad), this means that $\omega_{\rm o} \ll \omega_3$ and hence $\omega = |\boldsymbol{\omega}| \approx \omega_3$. Similarly from (10.95), $L = |\mathbf{L}| \approx L_3 = \lambda_3\omega_3$. Therefore the rate of precession of $\boldsymbol{\omega}$ in the space frame, as given by Eq.(10.96), is $\Omega_{\rm s} = L/\lambda_1 \approx \lambda_3\omega_3/\lambda_1 \approx \omega_3$, and the corresponding period is $\tau_{\rm s} = 2\pi/\Omega_{\rm s} \approx 2\pi/\omega_3 = 1$ day.

10.48 ⋆⋆ **(a)** Starting from Eq.(10.97), we get (see Fig.10.10 to check the expressions for the various unit vectors)

$$\begin{aligned}\boldsymbol{\omega} &= \dot\phi\hat{\mathbf{z}} + \dot\theta\mathbf{e}_2' + \dot\psi\mathbf{e}_3 \\ &= \dot\phi\hat{\mathbf{z}} + \dot\theta(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}) + \dot\psi[\cos\theta\hat{\mathbf{z}} + \sin\theta(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}})] \\ &= (-\dot\theta\sin\phi + \dot\psi\sin\theta\cos\phi)\hat{\mathbf{x}} + (\dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi)\hat{\mathbf{y}} + (\dot\phi + \dot\psi\cos\theta)\hat{\mathbf{z}}.\end{aligned}$$

(b) Starting from Eq.(10.99), we get

$$\begin{aligned}\boldsymbol{\omega} &= (-\dot\phi\sin\theta)\mathbf{e}_1' + \dot\theta\mathbf{e}_2' + (\dot\psi + \dot\phi\cos\theta)\mathbf{e}_3 \\ &= (-\dot\phi\sin\theta)(\cos\psi\mathbf{e}_1 - \sin\psi\mathbf{e}_2) + \dot\theta(\sin\psi\mathbf{e}_1 + \cos\psi\mathbf{e}_2) + (\dot\psi + \dot\phi\cos\theta)\mathbf{e}_3 \\ &= (-\dot\phi\sin\theta\cos\psi + \dot\theta\sin\psi)\mathbf{e}_1 + (\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)\mathbf{e}_2 + (\dot\psi + \dot\phi\cos\theta)\mathbf{e}_3\end{aligned}$$

10.52 ⋆⋆ **(a)** If θ is constant ($\dot\theta = 0$), the expression (10.100) for $\mathbf{L}$ simplifies to

$$\mathbf{L} = -\lambda_1\dot\phi\sin\theta\,\mathbf{e}_1' + L_3\,\mathbf{e}_3$$

where I have used (10.101) to replace $\lambda_3(\dot\psi + \dot\phi\cos\theta)$ by L_3. Next, using Fig.10.10, I shall read off the horizontal component $L_{\rm hor}$ of $\mathbf{L}$ and then replace $\dot\phi$ by the value found in (10.112) for the fast steady precession:

$$L_{\rm hor} = -\lambda_1\dot\phi\sin\theta\cos\theta + L_3\sin\theta = -\lambda_1\frac{L_3}{\lambda_1\cos\theta}\sin\theta\cos\theta + L_3\sin\theta = 0$$

That is, $\mathbf{L}$ is in the vertical direction. (Since the value used for $\dot\phi$ is only approximate, the same is true of this result — $\mathbf{L}$ is close to the vertical.)

(b) Because $\mathbf{L}$ is vertical, θ is the angle between $\mathbf{L}$ and $\mathbf{e}_3$, so $L_3 = L\cos\theta$, and (10.112) becomes $\Omega = L_3/(\lambda_1\cos\theta) = L/\lambda_1$, which is the same as the rate $\Omega_{\rm s}$ found in (10.96).

10.56 ⋆⋆⋆ **(a)** Because $E = \frac{1}{2}\lambda_1\dot{\theta}^2 + U_{\text{eff}}(\theta)$, it is clear that at no time can $U_{\text{eff}}(\theta)$ exceed E. Now, if you look carefully at (10.114), you will see that, because of the factor of $\sin^2\theta$ in the denominator, $U_{\text{eff}}(\theta) \to \infty$ as $\theta \to 0$, *unless* $L_3 = L_z$. Therefore, unless $L_3 = L_z$ the top cannot pass through the position $\theta = 0$, and, conversely, if the top does visit $\theta = 0$, L_3 and L_z must be equal.

(b) Setting $L_3 = L_z = \lambda_3\omega_3$ in (10.114) we find that

$$\begin{aligned} U_{\text{eff}}(\theta) &= \frac{(\lambda_3\omega_3)^2(1-\cos\theta)^2}{2\lambda_1\sin^2\theta} + MgR\cos\theta + \text{const} \\ &\approx \frac{(\lambda_3\omega_3)^2(\theta^2/2)^2}{2\lambda_1\theta^2} - MgR\frac{\theta^2}{2} + \text{const} = \frac{1}{8}\left(\frac{(\lambda_3\omega_3)^2}{\lambda_1} - 4MgR\right)\theta^2 + \text{const} \end{aligned}$$

where, in passing to the second line, I expanded everything through order θ^2.

(c) Oscillations of θ about $\theta = 0$ will be stable if and only if the coefficient of θ^2 in $U_{\text{eff}}(\theta)$ is positive. Thus if $\omega_3 > 2\sqrt{MgR\lambda_1/\lambda_3^2} = \omega_{\min}$ the equilibrium at $\theta = 0$ will be stable. If $\omega_3 < \omega_{\min}$, it is unstable.