8.2 ⋆⋆ **(a)** The Lagrangian is $\mathcal{L} = T - U$ or

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}m_1\dot{\mathbf{r}}_1^{\,2} + \tfrac{1}{2}m_2\dot{\mathbf{r}}_2^{\,2} - \left[m_1 g z_1 + m_2 g z_2 + U(r)\right] \\ &= \left[\tfrac{1}{2}M\dot{\mathbf{R}}^2 - MgZ\right] + \left[\tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - U(\mathbf{r})\right] = \mathcal{L}_{\rm cm} + \mathcal{L}_{\rm rel},\end{aligned}$$

where I have chosen rectangular coordinates with z measured vertically up and Z is the z coordinate of the CM position, $Z = (m_1 z_1 + m_2 z_2)/M$.

(b) The Lagrange equations for the three components of $\mathbf{R}$ are

$$M\ddot{X} = 0, \qquad M\ddot{Y} = 0, \qquad M\ddot{Z} = -Mg,$$

so the CM moves just like a projectile of mass M. The Lagrange equations for the relative coordinates are

$$\mu\ddot{\mathbf{r}} = -\boldsymbol{\nabla}_{\mathbf{r}}\, U(r)$$

where $\boldsymbol{\nabla}_{\mathbf{r}}$ denotes the gradient with respect to the relative coordinates. This last equation is precisely Newton's second law for the motion of a single particle of mass μ, position $\mathbf{r}$, and potential energy $U(r)$.

8.4 ⋆ The x equation is

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \qquad \text{or} \qquad -\frac{\partial U}{\partial x} = \mu\ddot{x}$$

with corresponding equations for y and z. These are precisely Newton's second law, $\mathbf{F} = \mu\ddot{\mathbf{r}}$, for a single particle of mass μ and position $\mathbf{r} = (x, y, z)$, subject to the force $\mathbf{F} = -\boldsymbol{\nabla}U$.

8.8 ⋆⋆ The Lagrangian is $\mathcal{L} = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{1}{2}kr^2$. The Lagrange equation for $\mathbf{R}$ is $\ddot{\mathbf{R}} = 0$, so the CM moves with constant velocity. The equation for $\mathbf{r}$ is $\mu\ddot{\mathbf{r}} = -k\mathbf{r}$, which implies that the relative position moves like a two-dimensional isotropic oscillator with angular frequency $\omega = \sqrt{k/\mu}$.

8.10 ⋆⋆ **(a)** The KE is given by (8.12), and the PE is just $U_1 + U_2 + U_{12}$. Using (8.9) you can check that

$$U = U_1 + U_2 + U_{12} = \tfrac{1}{2}k({r_1}^2 + {r_2}^2) + \tfrac{1}{2}\alpha k r^2 = k\mathbf{R}^2 + \tfrac{1}{2}k(\alpha + \tfrac{1}{2})\mathbf{r}^2.$$

(In deriving the last expression, remember that $m_1 = m_2$ so that $\mathbf{r}_1$ and $\mathbf{r}_2$ are just $\mathbf{R} \pm \mathbf{r}/2$. Also, $M = 2m_1$ and the reduced mass is $\mu = \frac{1}{2}m_1$.) Therefore

$$\mathcal{L} = \tfrac{1}{2}M\dot{\mathbf{R}}^2 + \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - k\mathbf{R}^2 - \tfrac{1}{2}k(\alpha + \tfrac{1}{2})\mathbf{r}^2$$

where $\mathbf{r}^2 = x^2 + y^2$, and so on.

(b) There are four Lagrange equations. That for the CM coordinate X reads

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} \qquad \text{or} \qquad -2kX = M\ddot{X}$$

with exactly the same equation for Y. Thus both components of the CM position oscillate with the same frequency $\sqrt{2k/M}$, and the CM moves around an elliptical path, as described in Section 5.3.

The equation for the relative coordinate x reads

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad \text{or} \qquad -k(\alpha + \tfrac{1}{2})x = \mu\ddot{x}$$

with exactly the same equation for y. Therefore both components of the relative position $\mathbf{r}$ oscillate with the same frequency $\sqrt{k(\alpha + \frac{1}{2})/\mu}$, and $\mathbf{r}$ also moves around an ellipse.

8.12 ⋆⋆ **(a)** According to Eq.(8.29), $\mu\ddot{r} = -dU_{\text{eff}}/dr$. Therefore, the planet can orbit at a fixed radius if and only if $dU_{\text{eff}}/dr = 0$. Since $U_{\text{eff}} = -\gamma/r + \ell^2/2\mu r^2$, it follows that $dU_{\text{eff}}/dr = \gamma/r^2 - \ell^2/\mu r^3$, which is zero when $r = r_\text{o} = \ell^2/\gamma\mu$.

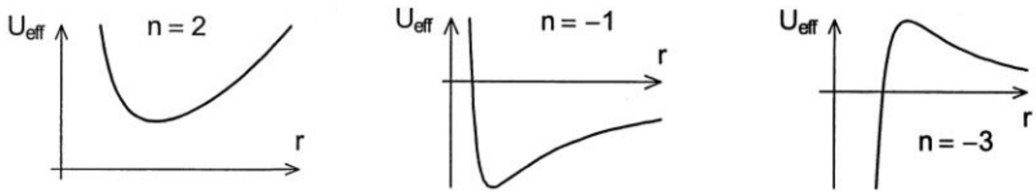
(b) The "equilibrium" radius r_o is stable if and only if U_{eff} is minimum at r_o; that is, its second derivative must be positive. This derivative is

$$\left[\frac{d^2U_{\text{eff}}}{dr^2}\right]_{r=r_\text{o}} = \left[\frac{-2\gamma}{r^3} + \frac{3\ell^2}{\mu r^4}\right]_{r=r_\text{o}} = \frac{\gamma}{r_\text{o}^3}$$

where, in the second equality, I used the result of part (a) to write $\ell^2 = \gamma\mu r_\text{o}$. Since this second derivative is positive, the equilibrium is stable. Near the minimum, the effective PE has the approximate form $U_{\text{eff}} \approx \text{const} + \frac{1}{2}(\gamma/{r_\text{o}}^3)(r - r_\text{o})^2$. Substituting this into the equation of motion, we get $\mu\ddot{r} = -dU_{\text{eff}}/dr = -(\gamma/r_\text{o}^3)(r - r_\text{o})$, which shows that r oscillates about r_o with angular frequency $\omega_{\text{osc}} = \sqrt{\gamma/\mu {r_\text{o}}^3}$, which is exactly the same as the angular velocity of the planet in its circular orbit. (To see this, set the centripetal acceleration $\omega^2 r$ equal to F_{grav}/μ.) Therefore, the period of oscillation is equal to the orbital period.

8.14 ⋆⋆⋆ **(a)** If $U = kr^n$, the force is $F = -dU/dr = -knr^{n-1}$. That $kn > 0$ means simply that the force is attractive (inward, toward the origin). See the graphs on the next page.

(b) An orbit of fixed radius r_o occurs if the derivative of U_{eff} is zero at $r = r_\text{o}$. The relevant functions are



$$U_{\text{eff}} = kr^n + \frac{\ell^2}{2\mu r^2}, \quad U'_{\text{eff}} = knr^{n-1} - \frac{\ell^2}{\mu r^3}, \quad \text{and} \quad U''_{\text{eff}} = kn(n-1)r^{n-2} + 3\frac{\ell^2}{\mu r^4}.$$

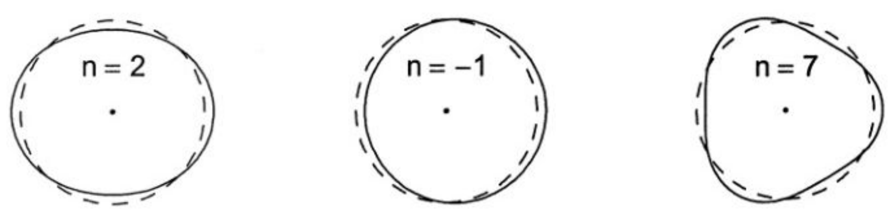
The derivative vanishes at radius $r_{\text{o}} = (\ell^2/\mu kn)^{1/(n+2)}$. The corresponding circular orbit is stable if the second derivative is positive at $r = r_{\text{o}}$. As you can check, after a little algebra, $U''_{\text{eff}}(r_{\text{o}}) = (n+2)\ell^2/\mu r_{\text{o}}^{\,4}$. Therefore the circular orbit is stable if $n > -2$. This agrees with the graphs where you can see that U_{eff} has a minimum for $n = 2$ and $n = -1$, but a maximum for $n = -3$.

(c) When $r = r_{\text{o}} + \epsilon$, with ϵ small, the effective PE is approximately

$$U_{\text{eff}}(r) = \text{const} + \tfrac{1}{2}U''_{\text{eff}}(r_{\text{o}})\epsilon^2 = \text{const} + \frac{1}{2}\frac{(n+2)\ell^2}{\mu r_{\text{o}}^{\,4}}\epsilon^2$$

and, provided $n > -2$, the equation of motion, $\mu\ddot{\epsilon} = -U'_{\text{eff}}$, implies radial oscillations of angular frequency $\omega_{\text{osc}} = \sqrt{n+2}\,\ell/\mu r_{\text{o}}^{\,2} = \sqrt{n+2}\,\omega$, where $\omega = \ell/\mu r_{\text{o}}^{\,2}$ is the angular velocity of the circular orbit. That is, $\tau_{\text{osc}} = \tau/\sqrt{n+2}$, as claimed.

If $\sqrt{n+2}$ is rational, $\sqrt{n+2} = p/q$ where p and q are integers, then after a time $t = p\tau_{\text{osc}} = q\tau$ both the orbital motion and the radial oscillations will be back where they started; that is, the whole motion will be about to repeat itself. In the pictures, the dashed circles show the circular orbits and the solid curves the motion with small radial oscillations.



8.16 $\star\star$ Multiplying both sides of the given equation by $(1+\epsilon\cos\phi)$ gives $r+\epsilon x = c$ (since $r\cos\phi = x$) or $r = c - \epsilon x$. Squaring both sides, setting $r^2 = x^2 + y^2$, and rearranging, we find $(1-\epsilon^2)x^2 + 2c\epsilon x + y^2 = c^2$. If we divide both sides by $(1-\epsilon^2)$ and define $d = c\epsilon/(1-\epsilon^2)$, this gives

$$(x^2 + 2dx) + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2}\,.$$

Next we can add d^2 to both sides to "complete the square" on the left, to give

$$(x+d)^2 + \frac{y^2}{1-\epsilon^2} \;=\; \frac{c^2}{1-\epsilon^2} + d^2 \;=\; \frac{c^2}{1-\epsilon^2}\left(1 + \frac{\epsilon^2}{1-\epsilon^2}\right) \;=\; \frac{c^2}{(1-\epsilon^2)^2} \;=\; a^2$$

if we define $a = c/(1-\epsilon^2)$. Finally, dividing through by a^2, we arrive at

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{a^2(1-\epsilon^2)} \;=\; \frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} \;=\; 1$$

where in the second expression I have introduced the definition $b = a\sqrt{1-\epsilon^2}$. Collecting our definitions of a, b, and d, we see that

$$a = \frac{c}{1-\epsilon^2}, \quad b = \frac{c}{\sqrt{1-\epsilon^2}} \quad \text{and} \quad d = a\epsilon$$

exactly as in (8.52).

8.17 $\star\star$ **(a)** If $G = \mathbf{r}\cdot\mathbf{p}$, then $dG/dt = \dot{\mathbf{r}}\cdot\mathbf{p} + \dot{\mathbf{p}}\cdot\mathbf{r} = mv^2 + \mathbf{F}\cdot\mathbf{r}$. If we integrate this from 0 to t, we get $G(t) - G(0) = \int_0^t (2T + \mathbf{F}\cdot\mathbf{r})dt$, or, dividing both sides by t,

$$\frac{G(t) - G(0)}{t} = 2\langle T\rangle + \langle \mathbf{F}\cdot\mathbf{r}\rangle. \tag{iii}$$

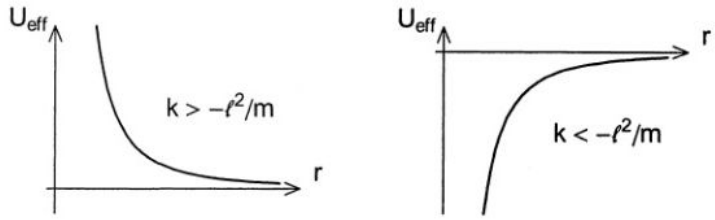
(b) If the motion is periodic and if K denotes the maximum value of $|G|$ during any one cycle, then the numerator of the left side of this relation can never exceed $2K$. Therefore, as we let $t\to\infty$, the left side approaches zero.

(c) If $U = kr^n$, then $\mathbf{F} = -\boldsymbol{\nabla}U = -nkr^{n-1}\hat{\mathbf{r}}$, so $\mathbf{F}\cdot\mathbf{r} = -nkr^n = -nU$. Inserting this in Eq.(iii) and letting $t\to\infty$, we find $0 = 2\langle T\rangle - \langle nU\rangle$, if we now understand the two angle brackets $\langle\,\rangle$ to denote the long-term average of whatever is between them. Therefore, $\langle T\rangle = n\langle U\rangle/2$.

8.18 $\star\star$ We are given the satellite's height $h_{\rm min} = 250$ km and speed $v_{\rm max} = 8500$ m/s at perigee. The distance from the earth's center is then $r_{\rm min} = R_{\rm e} + h_{\rm min} = 6650$ km. For any known satellite, we can certainly ignore the difference between the mass m and the reduced mass $\mu \approx m$. Thus the angular momentum is $\ell = mv_{\rm max}r_{\rm min}$ and the parameter c of Eq.(8.48) is $c = \ell^2/\gamma\mu = (v_{\rm max}r_{\rm min})^2/GM_{\rm e}$. (Recall that $\gamma = GM_{\rm e}m$.) Putting in the given numbers, we get $c = 7960$ km. The rest is easy: From Eq.(8.50), $r_{\rm min} = c/(1+\epsilon)$, so $\epsilon = (c - r_{\rm min})/r_{\rm min} = 0.197$. Similarly, from Eq.(8.50) $r_{\rm max} = c/(1-\epsilon) = 9910$ km, so $h_{\rm max} = r_{\rm max} - R_{\rm e} = 3510$ km.

8.22 ⋆⋆⋆ **(a)** If $F = k/r^3$, then $U = k/2r^2$, and the effective potential energy is

$$U_{\text{eff}} = U + \frac{\ell^2}{2mr^2} = \frac{k + \ell^2/m}{2r^2}.$$



If $k > -\ell^2/m$, the effective PE is positive (left picture) and the particle can come in from afar but must eventually move out to infinity. If $k < -\ell^2/m$, the effective PE is negative (right picture); if $E > 0$ the particle can come in from afar and then move out again, but if $E < 0$ it is trapped in a bounded orbit.

(b) The transformed equation reads $u'' = -(1 + km/\ell^2)u$. If $k > -\ell^2/m$, the number in parentheses is positive (call it κ^2) and the general solution is $u(\phi) = A\cos(\kappa\phi - \delta)$. By conservation of angular momentum, the angle ϕ always changes in one direction (always increasing or always decreasing). Therefore, the factor $\cos(\kappa\phi - \delta)$ must eventually vanish, so that $u \to 0$ and hence $r \to \infty$; that is, the particle eventually moves off to infinity, as predicted.

If $k < -\ell^2/m$, the transformed equation has the form $u'' = \lambda^2 u$, with the general solution $u(\phi) = Ae^{\lambda\phi} + Be^{-\lambda\phi}$. This solution may or may not vanish, depending on the values of A and B. If it vanishes, then r moves off to infinity at some value of ϕ. (In this case $E \geq 0$.) If $u(\phi)$ remains bounded away from zero ($u \geq u_{\text{o}}$ for some $u_{\text{o}} > 0$), then r remains bounded and the particle stays within some $r_{\max}$ at all times. (This is the case that $E < 0$.)

8.24 ⋆⋆⋆ With $\lambda < 0$ and $\ell^2 < -\lambda m$, the transformed equation for $u = 1/r$ can be written as

$$u''(\phi) = \left(\frac{m|\lambda|}{\ell^2} - 1\right)u + \frac{mk}{\ell^2} = \kappa^2(u + K)$$

with the general solution $u(\phi) = -K + Ae^{\kappa\phi} + Be^{-\kappa\phi}$. It is easy to see that this function can vanish no more than twice. Thus there are really just two cases to consider: (Remember that, because angular momentum is conserved, ϕ always increases or always decreases. To be definite, let's assume ϕ always increases. Remember also that by definition r and $u = 1/r$ are positive.)

(1) In the range $\phi_{(t=0)} \leq \phi \leq \infty$, the function $u(\phi)$ never vanishes, so that $u(\phi) \geq u_{\min} > 0$. In this case $r \leq r_{\max}$; that is, r is bounded. As $\phi \to \infty$, the function $u(\phi)$ approaches $Ae^{\kappa\phi}$ which approaches infinity. That is, $r \to 0$ and the particle eventually spirals in toward the origin. (As you can check, A cannot be zero in this case.)

(2) In the range $\phi_{(t=0)} \leq \phi \leq \infty$, the function u vanishes at least once, and the first time it does so is at $\phi = \phi_{\text{o}}$ (where ϕ_{o} may be infinity). In this case, as $\phi \to \phi_{\text{o}}$, $u(\phi)$ decreases toward 0 and $r = 1/u$ increases toward infinity; that is, the particle spirals out toward $r = \infty$.

8.26 ⋆⋆⋆ We have seen that Kepler's second law ("equal areas in equal times") is equivalent to conservation of angular momentum, which in turn implies that the force is central. Since the force is central and conservative, the variable $u = 1/r$ satisfies the "transformed radial equation" (8.41), which we can rewrite as

$$F = -[u''(\phi) + u(\phi)]\ell^2 u(\phi)^2/\mu.$$

Next, Kepler's first law states that the path of any body orbiting the sun is an ellipse with the sun at one focus, and we have seen that the equation for such an ellipse has the form (8.49), namely, $u(\phi) = (1 + \epsilon\cos\phi)/c$, where ϵ and c are positive constants for any given ellipse. When this form is substituted into the transformed radial equation, we find that

$$F = -\frac{\ell^2 u^2}{c\mu} = -\frac{\ell^2/(c\mu)}{r^2}.$$

Finally, because the force is conservative, it cannot depend on the angular momentum of the body, and the constant c must be proportional to ℓ^2, and we're left with $F = -\gamma/r^2$ where γ is a positive constant.

8.30 ⋆⋆ If we multiply both sides of Eq.(8.49), $r = c/(1 + \epsilon\cos\phi)$ by $(1 + \epsilon\cos\phi)$, replace $r\cos\phi$ by x, and rearrange, we find that $r = c - \epsilon x$. Squaring both sides gives $x^2 + y^2 = c^2 - 2c\epsilon x + \epsilon^2 x^2$. We now have two cases to consider. **(a)** If $\epsilon = 1$, the terms in x^2 cancel and we're left with $y^2 = c^2 - 2cx$, a parabola. **(b)** If $\epsilon > 1$, we find $(\epsilon^2 - 1)x^2 - 2c\epsilon x - y^2 = -c^2$. Completing the square for x gives $(\epsilon^2 - 1)(x - \delta)^2 - y^2 = -c^2 + \epsilon^2 c^2/(\epsilon^2 - 1) = c^2/(\epsilon^2 - 1)$. Finally, multiplying both sides by $(\epsilon^2 - 1)/c^2$, we get

$$\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1 \quad \text{where} \quad \alpha = \frac{c}{\epsilon^2 - 1},\ \beta = \frac{c}{\sqrt{\epsilon^2 - 1}} \text{ and } \delta = \epsilon\alpha$$

which is the equation of a hyperbola.

8.34 ⋆⋆ If we use the notation of Example 8.6, $R_1 = 1$ AU and $R_3 = 30$ AU. Therefore the semi-major axis of the transfer orbit is $a_2 = (R_1 + R_3)/2 = 15.5$ AU and the period of the transfer orbit is $\tau_2 = \tau_e(a_2/a_e)^{3/2} = \tau_e(15.5)^{3/2}$. The time for the transfer is half of this period, namely $\frac{1}{2}\tau_2 = \frac{1}{2}\tau_e(15.5)^{3/2} = 30.5$ years.
