Physics 304 Spring 2021 Exam 3 NAME: Solutions

1) A bead of mass m is threaded on a frictionless wire that is bent into the shape of a helix with cylindrical polar coordinates (ρ, φ, z) satisfying z = cφ and ρ = R, with c and R constants. The z axis points vertically up and gravity vertically down. Use φ as your generalized coordinate and follow the steps discussed in class to solve Hamilton's equations.

The kinetic energy is $T = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2$, ($\dot{\rho}^2 \to 0$, since confined to cylinder), with $\rho = R \Rightarrow$

$$T = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2) \quad \text{but } z = c\phi \Rightarrow \dot{z} = c\dot{\phi} \Rightarrow$$

$$T = \frac{1}{2}m(R^2\dot{\phi}^2 + c^2\dot{\phi}^2) = \frac{1}{2}m(R^2+c^2)\dot{\phi}^2$$

and $U = mgz = mgc\phi$

$$p = \frac{\partial T}{\partial \dot{\phi}} = m(R^2+c^2)\dot{\phi}$$

$$\therefore \mathcal{H} = T + U = \frac{p^2}{2m(R^2+c^2)} + mgc\phi$$

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m(R^2+c^2)} \quad \text{and} \quad \dot{p} = \frac{-\partial \mathcal{H}}{\partial \phi} = -mgc$$

$$\Downarrow$$

$$p = m(R^2+c^2)\dot{\phi} \Rightarrow \dot{\phi} = \frac{\dot{p}}{m(R^2+c^2)} = \frac{1}{c}\ddot{z} \Rightarrow$$

$$\ddot{z} = \frac{c\dot{p}}{m(R^2+c^2)} = -\frac{c(mgc)}{m(R^2+c^2)} = -\frac{gc^2}{(R^2+c^2)}$$

2) The stress tensor of some solid is given by:

$$\Sigma = \begin{bmatrix} x^2 & -2xy & 3z \\ -2xy & 2xy & y^2 \\ 3z & y^2 & -1 \end{bmatrix}$$ Find the surface force on a small area dA of the surface $2x + y^2 + 3z = 2$ at the point (1, 0, 3).

$\vec{n} = \vec{\nabla} f = 2\hat{x} + 2y\hat{y} + 3\hat{z}$

at $(1,0,3)$, $\vec{n} = 2\hat{x} + 3\hat{z}$, so $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

$$\vec{F} = \iint \hat{n}\, dA = \frac{dA}{\sqrt{13}} \begin{bmatrix} 1 & 0 & 9 \\ 0 & 0 & 0 \\ 9 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \frac{dA}{\sqrt{13}} \begin{bmatrix} 29 \\ 0 \\ 15 \end{bmatrix}$$

not in the same direction as $\hat{n}$