Physics 304 Spring 2023 Exam 2 NAME: Solutions

1) An inertia tensor is calculated in one frame (x,y,z) to be:

$$md^2 \begin{bmatrix} 8 & -4 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix}.$$ Find the principal moments and principal axes.

$md^2 = \mu$

$$\begin{vmatrix} 8\mu-\lambda & -4\mu & 0 \\ -4\mu & 2\mu-\lambda & 0 \\ 0 & 0 & 10\mu-\lambda \end{vmatrix} = (8\mu-\lambda)(2\mu-\lambda)(10\mu-\lambda) - 16\mu^2(10\mu-\lambda)$$

$$= (10\mu-\lambda)\left[(2\mu-\lambda)(8\mu-\lambda) - (16\mu^2)\right] = 0 \therefore \lambda_1 = 10\mu$$

also, $(2\mu-\lambda)(8\mu-\lambda) - 16\mu^2 = 16\mu^2 - 10\mu\lambda + \lambda^2 - 16\mu^2 = 0 \Rightarrow$

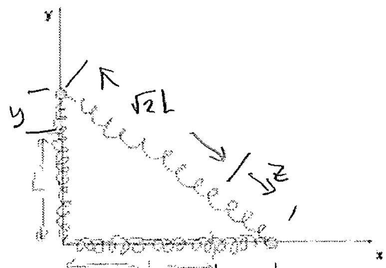
$$10\mu\lambda - \lambda^2 = 0 \Rightarrow \lambda_2 = 0,\ \lambda_3 = 10\mu$$

for $\lambda_2 = 0$, $\begin{bmatrix} 8\mu & -4\mu & 0 \\ -4\mu & 2\mu & 0 \\ 0 & 0 & 10\mu \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \Rightarrow$ $4\mu(2\omega_x - \omega_y) = 0$, $2\mu(-2\omega_x + \omega_y) = 0$, $10\mu\omega_z = 0$ $\Rightarrow \hat{e}_1 = \frac{1}{\sqrt{5}}(1,2,0)$

for $\lambda_1 = \lambda_3 = 10\mu$, $\begin{bmatrix} -2\mu & -4\mu & 0 \\ -4\mu & -8\mu & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \Rightarrow$ $-2\mu(\omega_x + 2\omega_y) = 0$, $-4\mu(\omega_x + 2\omega_y) = 0$ $\Rightarrow \hat{e}_2 = \frac{1}{\sqrt{5}}(-2,1,0)$

and $\hat{e}_3 = (0,0,1)$

2) Two equal masses are constrained to move without friction on the positive x and y axes. They are attached to two identical springs (force constant k) whose other ends are attached to the origin. In addition, the masses are connected to each other by another spring with the same force constant, k. The system is in equilibrium when all three springs are in their relaxed state, $L_1 = L_2 = L$, and $L_3 =$ sqrt(2)*L. Find the normal frequencies and modes for small oscillations about equilibrium. There is no gravity.



$$U = \tfrac{1}{2}k(x^2 + y^2) + \tfrac{1}{2}kz^2$$

where $z = \sqrt{x^2 + 2Lx + L^2 + y^2 + 2Ly + L^2} - \sqrt{2}L$

drop terms in x^2 & y^2 (small osc.)

$$z = \sqrt{2L(x+y) + 2L^2} - \sqrt{2}L$$

$$= \sqrt{2}L\left(\sqrt{1 + \tfrac{x+y}{L}} - 1\right)$$

$$\approx \sqrt{2}\cancel{L}\left(\cancel{1} + \tfrac{1}{2}\tfrac{(x+y)}{\cancel{L}} - \cancel{1}\right)$$

$$\approx \frac{x+y}{\sqrt{2}}$$

$$\therefore U = \tfrac{1}{2}k(x^2 + y^2) + \tfrac{1}{4}k(x+y)^2$$

$$= \tfrac{1}{2}k\left(\tfrac{3}{2}x^2 + \tfrac{3}{2}y^2\right) + \tfrac{1}{2}kxy$$

$$\therefore \mathbb{K} = \begin{bmatrix} \tfrac{3}{2}k & \tfrac{1}{2}k \\ \tfrac{1}{2}k & \tfrac{3}{2}k \end{bmatrix}$$

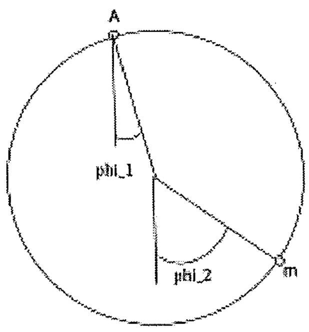
$$\therefore \begin{vmatrix} \tfrac{3}{2}k - m\omega^2 & \tfrac{1}{2}k \\ \tfrac{1}{2}k & \tfrac{3}{2}k - m\omega^2 \end{vmatrix} = 0 \Rightarrow$$

$$\begin{vmatrix} \tfrac{3}{2}k - m\omega^2 & -k + m\omega^2 \\ \tfrac{1}{2}k & k - m\omega^2 \end{vmatrix} = \begin{vmatrix} 2k - m\omega^2 & 0 \\ \tfrac{1}{2}k & k - m\omega^2 \end{vmatrix} = (2k - m\omega^2)(k - m\omega^2) = 0 \Rightarrow \omega_1^2 = \frac{k}{m}, \ \omega_2^2 = \frac{2k}{m}$$

for ω_1, $\begin{bmatrix} \tfrac{1}{2}k & \tfrac{1}{2}k \\ \tfrac{1}{2}k & \tfrac{1}{2}k \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow a_1 = -a_2$ and the two masses osc. with equal amplitudes, but opposite phases

for ω_2, $\begin{bmatrix} -\tfrac{1}{2}k & \tfrac{1}{2}k \\ \tfrac{1}{2}k & -\tfrac{1}{2}k \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow a_1 = a_2$ and the two masses osc. with equal amplitudes and in phase

3) A bead of mass m is threaded on a frictionless wire hoop of radius R and mass m (same mass). The hoop is attached to a wall at point A on a frictionless pivot and can rotate about this point in the plane of the hoop. Find the normal modes and frequencies for small oscillations.



(parallel axis theorem)

$$I^{CM}_{Hoop} = mR^2 \therefore I^{A}_{Hoop} = mR^2 + mR^2 = 2mR^2$$

$$\therefore T_{Hoop} = \frac{1}{2} I\omega^2 = \frac{1}{2}(2mR^2)\dot{\phi}_1^2 = mR^2\dot{\phi}_1^2$$

0 for small osc.

$$T_{mass} = \frac{1}{2}mv^2 = \frac{1}{2}m\left(R\dot{\phi}_1^2 + R\dot{\phi}_2^2 + 2R\cos(\phi_1 - \phi_2)\dot{\phi}_1\dot{\phi}_2\right)$$

$$\therefore T = \frac{1}{2}mR^2(3\dot{\phi}_1^2 + 2\dot{\phi}_1\dot{\phi}_2 + \dot{\phi}_2^2)$$

$$U_{Hoop} = mgR(1-\cos\phi_1) \approx \frac{1}{2}mgR\phi_1^2$$

$$U_{mass} = mgR[(1-\cos\phi_1) + (1-\cos\phi_2)] \approx \frac{1}{2}mgR\phi_1^2 + \frac{1}{2}mgR\phi_2^2$$

$$\therefore U = \frac{1}{2}mgR(2\phi_1^2 + \phi_2^2)$$

$$\therefore \mathbb{M} = \frac{1}{2}mR^2\begin{pmatrix}3 & 1\\ 1 & 1\end{pmatrix}, \quad \mathbb{K} = \frac{1}{2}mgR\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{2}mR^2\omega_0^2\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}, \quad \omega_0 = \sqrt{g/R}$$

-2×2^{nd} row + 1^{st} row; 1^{st} col. + 2^{nd} col

$$\therefore \begin{vmatrix}2\omega_0^2 - 3\omega^2 & -\omega^2\\ -\omega^2 & \omega_0^2 - \omega^2\end{vmatrix} = 0 = \begin{vmatrix}2\omega_0^2 - \omega^2 & -2\omega_0^2 + \omega^2\\ -\omega^2 & \omega_0^2 - \omega^2\end{vmatrix} = \begin{vmatrix}2\omega_0^2 - \omega^2 & 0\\ -\omega^2 & \omega_0^2 - 2\omega^2\end{vmatrix}$$

$$\therefore (2\omega_0^2 - \omega^2)(\omega_0^2 - 2\omega^2) = 0 \Rightarrow \omega_1^2 = \frac{1}{2}\omega_0^2, \quad \omega_2^2 = 2\omega_0^2$$

$$\text{for } \omega_1, \begin{bmatrix}\frac{1}{2}\omega_0^2 & -\frac{1}{2}\omega_0^2\\ -\frac{1}{2}\omega_0^2 & \frac{1}{2}\omega_0^2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} = 0 \Rightarrow a_1 = a_2$$

$$\text{for } \omega_2, \begin{bmatrix}-4\omega_0^2 & -2\omega_0^2\\ -2\omega_0^2 & -1\omega_0^2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} = 0 \Rightarrow a_1 = -\frac{1}{2}a_2$$