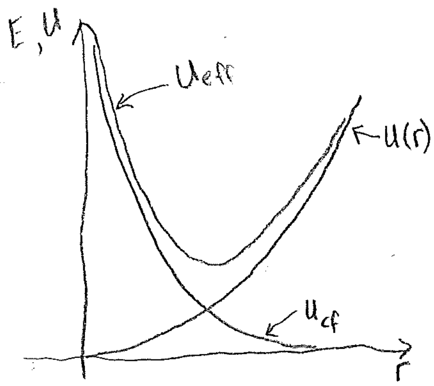
Physics 304 Spring 2023 Exam 1 NAME: Solutions

1) For a potential energy between two particles of reduced mass μ of $U(r) = \frac{1}{2}kr^2$, $(k>0)$ and angular momentum ℓ, make a sketch of the potential, the centrifugal potential, and the effective potential (all on the same graph). Find the value of r where the two particles remain at a constant distance, and find the frequency of small oscillations if they are disturbed from their equilibrium position.



$$U_{cf} = \frac{\ell^2}{2\mu r^2}$$

$$U_{eff} = \frac{1}{2}kr^2 + \frac{\ell^2}{2\mu r^2}$$

$$\left.\frac{\partial U_{eff}}{\partial r}\right|_{equil.} = kr - \frac{\ell^2}{\mu r^3} = 0 \Rightarrow$$

$$r_{equil}^4 = \frac{\ell^2}{\mu k} \Rightarrow r_{equil} = \left(\frac{\ell^2}{\mu k}\right)^{1/4}$$

Taylor expand U_{eff} about $r_{equil.}$:

$$U_{eff} = U_{eff}(r_{eq}) + U'_{eff}(r_{eq})(r-r_{eq}) + \frac{1}{2}U''_{eff}(r_{eq})(r-r_{eq})^2 + \ldots$$

$$\Rightarrow \omega^2 = \frac{U''_{eff}(r_{eq})}{\mu}, \quad U''_{eff}(r_{eq}) = k + \frac{3\ell^2}{\mu r_{eq}^4} = k + \frac{3\ell^2}{\mu\left(\frac{\ell^2}{\mu k}\right)} = 4k$$

$$\therefore \omega^2 = \frac{4k}{\mu}$$

$$\omega = 2\sqrt{\frac{k}{\mu}}$$

2) A satellite ($m=1x10^5$kg) of the sun ($M_S=1.99x10^{30}$kg) is found at a distance from the sun of $1x10^{11}$m, with a speed of $1x10^5$m/s, with an angle between the vector from the sun to the satellite and the velocity vector of 70°. What is the distance of closest and farthest approach of the satellite to the sun, and what is the eccentricity of the orbit.

The satellite's angular momentum is $|\vec{\ell}| = |\vec{r} \times \vec{p}| = rmv\sin\theta$

$$\therefore \ell = (1\times10^{11}\,m)(1\times10^{5}\,kg)(1\times10^{5}\,\tfrac{m}{s})\cdot \sin 70^\circ$$

$$= 9.4\times10^{20}\,\frac{kg\,m^2}{s}$$

The total energy of the satellite is $E = T + U$

$$E = \frac{1}{2}mv^2 - \frac{GM_S m}{r}$$

$$= \frac{1}{2}(1\times10^5\,kg)(1\times10^5\,\tfrac{m}{s})^2 - \frac{(6.67\times10^{-11}\,\frac{N\cdot m^2}{kg^2})(1.99\times10^{30}\,kg)(1\times10^5\,kg)}{1\times10^{11}\,m}$$

$$= 5\times10^{14}\,J - 1.33\times10^{14}\,J$$

$$= 3.67\times10^{14}\,J$$

From Eq. 8.58, $E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2-1) = \frac{(GM_s m)^2 m}{2\ell^2}(\epsilon^2-1) \Rightarrow$

$$\epsilon^2 = \frac{E\,2\ell^2}{(GM_s m)^2 m} + 1 \Rightarrow$$

$$\boxed{\epsilon = 6.067}$$

and $c = \frac{\ell^2}{GM_s m^2} \Rightarrow$

$$c = 6.657\times10^{11}\,m$$

$$\therefore r_{min} = \frac{c}{\epsilon+1} \Rightarrow r_{min} = 9.42\times10^{10}\,m$$

since $\epsilon > 1$, $r_{max} = \infty$ (hyperbola)

3) A puck is on a frictionless, horizontal, rotating table (rotating counterclockwise with angular velocity Ω) and is given an initial velocity $\mathbf{v_0} = (v_{0x}, v_{0y})$ from the origin (axis of rotation) both in the frame of an observer fixed on the table. Use Newton's laws to write the equations of motion in the observer's frame and solve.

Because we have no friction, and the gravitational and normal forces approximately cancel, we are left with only:

$$\vec{F}_{net} = 2m\dot{\vec{r}}\times\vec{\Omega} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = m\ddot{\vec{r}}$$

$$= 2m(\dot{x}\hat{x}+\dot{y}\hat{y})\times\Omega\hat{z} - m\Omega\hat{z}\times(\Omega\hat{z}\times(x\hat{x}+y\hat{y}))$$

$$= 2m(-\dot{x}\hat{y}+\dot{y}\hat{x})\Omega - m\Omega\hat{z}\times(\Omega x\hat{y}-\Omega y\hat{x})$$

$$= 2m\Omega(-\dot{x}\hat{y}+\dot{y}\hat{x}) + m\Omega^2(x\hat{x}+y\hat{y}) = m(\ddot{x}\hat{x}+\ddot{y}\hat{y}) \Rightarrow$$

① $m\ddot{x} - 2m\Omega\dot{y} - m\Omega^2 x = 0$

② & $m\ddot{y} + 2m\Omega\dot{x} - m\Omega^2 y = 0$

let $\eta = x+iy$, multiply ② by i and add

$$(\ddot{x}+i\ddot{y}) + 2\Omega i(\dot{x}+i\dot{y}) - \Omega^2(x+iy) = 0 \Rightarrow$$

$$\ddot{\eta} + 2\Omega i\dot{\eta} - \Omega^2\eta = 0 \Rightarrow \eta = e^{-i\alpha t} \Rightarrow$$

$$\alpha^2 - 2\Omega\alpha + \Omega^2 = 0 \Rightarrow$$

$$(\alpha-\Omega)^2 = 0 \Rightarrow \alpha = \Omega$$

$\therefore$ one solution is $\eta = e^{-i\Omega t}$, second is $\eta = te^{-i\Omega t}$

$$\therefore \eta = e^{-i\Omega t}(C_1 + C_2 t)$$

with $\eta(t=0) = 0 + i0 = C_1 = 0$

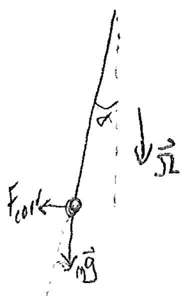
$$\dot{\eta}(t=0) = V_{0x} + iV_{0y} = -i\Omega e^{-i\Omega 0}(\cancel{C_2 t})^{0} + e^{-i\Omega\cdot 0}C_2 = C_2$$

$$\therefore \eta = (V_{0x}t + iV_{0y}t)(\cos\Omega t + i\sin\Omega t)$$

$\therefore$ $x(t) = Re(\eta) = V_{0x}t\cos\Omega t + V_{0y}t\sin\Omega t$

$y(t) = Im(\eta) = V_{0y}t\cos\Omega t - V_{0y}t\sin\Omega t$

4) A train is traveling at 200 m/s on a straight, horizontal track across the South Pole. Find the angle of a plumb line suspended from the ceiling inside the train and another inside a hut on the ground. In what direction is the plumb line on the train deflected?



The angle of the bob on the train will be determined by

$$\tan\alpha = \frac{F_{corr}}{mg}$$

$$= \frac{2v\Omega}{g} \quad , \quad \Omega = \frac{2\pi}{86{,}400}$$

$\therefore \ \alpha = 0{,}17°$ to the left.