Physics 303 Fall 2022 Exam 4   NAME: Solutions

SHOW ALL WORK

1) Find the magnitude of the incremental path, ds, of a particle moving on the surface of a sphere in terms of just **one** of the three spherical coordinate displacements (dr, dθ, dφ) and whatever other coordinate functions are needed.

$$d\vec{s} = (dr, r\,d\theta, r\sin\theta\,d\phi)$$

$$\therefore\ ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2$$

but, since it moves on the surface of a sphere of radius R,

$$dr = 0,\quad r = R$$

$$\therefore\ ds^2 = R^2 d\theta^2 + R^2\sin^2\theta\,d\phi^2$$

let $d\phi = \frac{d\phi}{d\theta}\cdot d\theta = \phi' d\theta$

then, $ds^2 = R^2 d\theta^2 + R^2\sin^2\theta\,\phi'^2 d\theta^2$

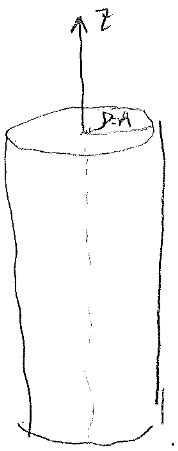
$$\therefore\ ds = R(1 + \sin^2\theta\,\phi'^2)^{1/2} d\theta$$

or: let $d\theta = \frac{d\theta}{d\phi} d\phi = \theta' d\phi$

then, $ds^2 = R^2\theta'^2 d\phi^2 + R^2\sin^2\theta\,d\phi^2$

$$\therefore\ ds = R(\theta'^2 + \sin^2\theta)\,d\phi$$

2) Find and describe the shortest path for two points on the surface of a cylinder using cylindrical polar coordinates (ρ,φ,z). Show all steps and sketch the result.



$$d\vec{s} = (d\rho,\ R\,d\phi,\ dz) \quad \text{but } d\rho = 0 \Rightarrow$$

$$(0,\ R\,d\phi,\ dz)$$

$$\therefore ds^2 = R^2 d\phi^2 + dz^2$$

$$= R^2\left(\frac{d\phi}{dz}\right)^2 dz^2 + dz^2 = (R^2\phi'^2 + 1)\,dz^2$$

$$\therefore ds = (R^2\phi'^2 + 1)^{1/2}\,dz$$

$$\therefore L = \int_{z_1}^{z_2} \underbrace{(R^2\phi'^2+1)^{1/2}}_{f}\,dz$$

$$\frac{\partial f}{\partial \phi} = 0 \Rightarrow \frac{\partial f}{\partial \phi'} = \text{const.} = \frac{R^2\phi'}{(R^2\phi'^2+1)^{1/2}} \Rightarrow$$

$$\left[C(R^2\phi'^2+1)^{1/2} = R^2\phi'\right]^2$$

$$C^2(R^2\phi'^2+1) = R^4\phi'^2$$

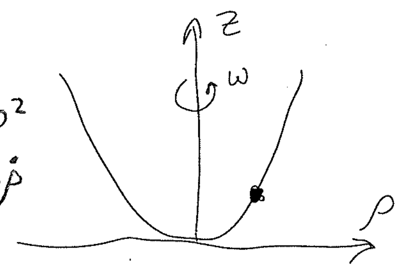
$$C^2 = (R^4 - C^2R^2)\,\phi'^2$$

$$\therefore \phi' = \text{const} = \frac{d\phi}{dz} \Rightarrow$$

$$\phi = mz + b$$

The path is described by the intersection of the cylinder with a plane containing the two points and the mid-point of the two points on the z-axis.

3) Consider a bead of mass m sliding without friction on a wire that is bent into the shape of a parabola and is being spun about its vertical axis as shown below. The equation of the parabola is $z=k\rho^2$. Find the equation of motion of the bead and determine points of equilibrium if any, and discuss stability.



In cylindrical coords, $\vec{v} = (\dot\rho, \dot z, \rho\dot\phi)$

$\therefore T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot\rho^2 + \dot z^2 + \rho^2\dot\phi^2)$, but $z = k\rho^2$

$\therefore \dot z = 2k\rho\dot\rho$

$\therefore T = \frac{1}{2}m(\dot\rho^2 + 4k^2\rho^2\dot\rho^2 + \rho^2\omega^2)$

and $U = mgz = mgk\rho^2$

$$\therefore \mathcal{L} = \frac{1}{2}m(\dot\rho^2 + \rho^2\omega^2 + 4k^2\rho^2\dot\rho^2) - mgk\rho^2$$

$$\frac{\partial\mathcal{L}}{\partial\rho} = m\rho\omega^2 + 4mk^2\rho\dot\rho^2 - 2mgk\rho$$

$$\frac{\partial\mathcal{L}}{\partial\dot\rho} = m\dot\rho + 4mk^2\rho^2\dot\rho$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\rho} = m\ddot\rho + 8mk^2\rho\dot\rho^2 + 4mk^2\rho^2\ddot\rho$$

$$\therefore (1+4k^2\rho^2)\ddot\rho + 4k^2\rho\dot\rho^2 - \rho\omega^2 + 2gk\rho = 0$$

To find equil. points, $\dot\rho = \ddot\rho = 0$ then

$(\omega^2 - 2gk)\rho = 0 \Rightarrow \rho = 0$ and $\omega^2 = 2gk$

at $\rho = 0$ with ρ and $\dot\rho$ small $\Rightarrow \ddot\rho \approx (\omega^2 - 2gk)\rho$

$\therefore$ stable if $\omega^2 < 2gk$

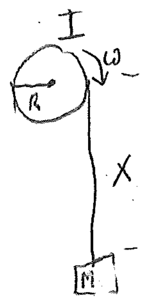
unstable if $\omega^2 > 2gk$

if $\omega^2 = 2gk$

equilibrium at <u>any</u> ρ, but unstable

since $\ddot\rho = -\frac{4k^2\rho\dot\rho^2}{(1+4k^2\rho^2)} = \frac{-\rho\dot\rho^2}{(\frac{1}{4k^2}+\rho^2)}$ always negative, but vanishes when $\dot\rho = 0$, so doesn't return to equil. point.

4) A string is wound around a frictionless pulley of radius R and moment of inertia I, and then a mass is hung from the string and released. Using the method of Lagrange multipliers, find the tension in the string and discuss limiting cases.



constraint eq.: $R\phi - x = \text{const.} \Rightarrow \ddot{x} = R\ddot{\phi}$ ③

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2$$

$$U = -mgx$$

$$\therefore \mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + mgx$$

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda\frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \Rightarrow$$

① $mg - \lambda = m\ddot{x}$

$$\frac{\partial \mathcal{L}}{\partial \phi} + \lambda\frac{\partial f}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \Rightarrow$$

② $\lambda R = I\ddot{\phi}$

③ into ① $\Rightarrow mg - \lambda = mR\ddot{\phi} = mR\frac{\lambda R}{I}$

$$\Rightarrow mg = \left(\frac{mR^2}{I} + 1\right)\lambda$$

$\therefore \lambda = \dfrac{mg}{\frac{mR^2}{I} + 1}$ and, since $T = F_c = \lambda\frac{\partial f}{\partial x}$

$$T = -\frac{mg}{\frac{mR^2}{I} + 1} = \frac{-mIg}{mR^2 + I}$$

if I is very large, $T = -mg$

if I is very small, $T = 0$