Lec 9: 09/14/2009

Now lets consider position and momentum operators in three dimensions. Similar to the x direction we have: ($P_x \equiv -i\hbar \frac{\partial}{\partial x}$, $P_y \equiv -i\hbar \frac{\partial}{\partial y}$, $P_z \equiv -i\hbar \frac{\partial}{\partial z}$)

$$[X, P_x] = [Y, P_y] = [Z, P_z] = i\hbar$$

$$[X, Y] = [X, Z] = [Y, Z] = 0$$

$$[P_x, P_y] = [P_x, P_z] = [P_y, P_z] = 0$$

$$[X, P_y] = [X, P_z] = [Y, P_x] = [Y, P_z] = [Z, P_x] = [Z, P_y] = 0$$

Therefore position and momentum along the same direction cannot be simultaneously measured (not compatible).

But position and/or momentum along different directions can be simultaneously measured (they are compatible).

Another important example is the angular momentum. Classically:

$\vec{L} = \vec{r} \times \vec{p} \Rightarrow L_x = y p_z - z p_y, \; L_y = z p_x - x p_z, \; L_z = x p_y - y p_x$

In quantome mechanics:

$L_x = Y P_z - Z P_y, \; L_y = Z P_x - X P_z, \; L_z = X P_y - Y P_x$

Note that since momentum and position along different directions commute, there will be no ambiguity that requires symmetrization. We have the following commotation relations among the components of the angular momentom vector $\vec{L}$:

$\vec{L} = L_x \hat{x} + L_y \hat{y} + L_z \hat{z}$

$[L_x, L_y] = i\hbar L_z \quad [L_y, L_z] = i\hbar L_x \quad [L_z, L_x] = i\hbar L_z$

Therefore different components of the angular momentum vector cannot be measured simultaneously (they are not compatible).

Now we define,

$L^2 = L_x^2 + L_y^2 + L_z^2$

And we have;

$[L_x, L^2] = [L_y, L^2] = [L_z, L^2] = 0$

Therefore L^2 and $L_x / L_y / L_z$ can be simultaneously measured. That is the total angular momentum and its x/y/z component are compatible. We will use this later on to label energy eigenstates of a system with spherically symmetric potential (like Hydrogen atom or three-dimesional harmonic oscillator).

Schrodinger and Heisenberg Pictures:

Propagator operates on the state vector of a system The state of the system is therefore time-dependent $|\Psi(t)\rangle$, while operators that represent observables are time-independent. This is called the Schrodinger picture. Propagator can be considered as an operator that transforms the state of the system, and other operators act on the transformed vector.

The Hamiltonian eigenbasis $|E\rangle$ is the same as a function of time, while the state vector is transformed. This is an active transformation:

$$|\Psi(t)\rangle = \sum_E a_E(0)\, e^{-i\frac{Et}{\hbar}} \underbrace{|E\rangle}_{\text{eigenbasis}}$$

But we can change the Hamiltonia eigenbasis such that the state vector is constant in time. The state vector then remains the same as a result of transformation of the Hamiltonian eigenbasis. This is a passive transformation.

In the instantaneous eigenbasis $|E(t)\rangle$, the state vector looks the same at all times:

$$|\psi\rangle = \sum_E a_E(0) \underbrace{e^{\frac{-iEt}{\hbar}} |E\rangle}_{|E(t)\rangle}$$

This is called the Heisenberg picture. In this picture a vector is related to the same vector in the Schrodinger picture according to:

$$|\psi\rangle_H = U^\dagger |\psi\rangle_S$$

Operators in the two pictures are related to each other according to:

$$\Omega_H = U^\dagger \Omega_S U$$

Note that $H_H = H_S$ since H commutes with U. For any operator that commutes with H, we have $\Omega_H = \Omega_S$. In the Heisenberg pictures the complete time dependence of the state vector is frozen, but operators are time-dependent in general.

An important point is that inner products, probabilities and expectation values are the same in both pictures:

$$ {}_H\langle \Psi_1 | \Psi_2 \rangle_H = {}_S\langle \Psi_1 | U U^\dagger | \Psi_2 \rangle_S = {}_S\langle \Psi_1 | \Psi_2 \rangle_S $$

$$ {}_H\langle \Psi | \Omega_H | \Psi \rangle_H = {}_S\langle \Psi | U U^\dagger \Omega_S U U^\dagger | \Psi \rangle_S = {}_S\langle \Psi | \Omega_S | \Psi \rangle_S $$