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Using Schrodinger equation, we can find time evolution of the expectation value of an observable.

$$\frac{d}{dt}\langle\Omega\rangle = \frac{d}{dt}\langle\Psi(t)|\Omega|\Psi(t)\rangle = \frac{d}{dt}\langle\Psi(0)|e^{\frac{iHt}{\hbar}}\Omega$$

$$e^{-\frac{iHt}{\hbar}}|\Psi(0)\rangle = \langle\Psi(0)|\left(\frac{d}{dt}e^{\frac{iHt}{\hbar}}\right)\Omega e^{-\frac{iHt}{\hbar}}|\Psi(0)\rangle +$$

$$\langle\Psi(0)|e^{\frac{iHt}{\hbar}}\Omega\left(\frac{d}{dt}e^{-\frac{iHt}{\hbar}}\right)|\Psi(0)\rangle = \langle\Psi(0)|e^{\frac{iHt}{\hbar}}\times\frac{iH}{\hbar}$$

$$\Omega e^{-\frac{iHt}{\hbar}}|\Psi(0)\rangle - \langle\Psi(0)|e^{\frac{iHt}{\hbar}}\Omega\frac{-iH}{\hbar}e^{-\frac{iHt}{\hbar}}|\Psi(0)\rangle$$

$$= \frac{i}{\hbar}\langle\Psi(t)|H\Omega - \Omega H|\Psi(t)\rangle = -\frac{i}{\hbar}\langle[\Omega, H]\rangle$$

Therefore, the expectation value of any observable whose corresponding operator commutes with the Hamiltonian is time independent.

Note that it may happen that $\langle\dot{\Omega}\rangle = 0$, even though $[\Omega, H] \neq 0$.

Example: Consider the following state vector;

$$\psi(x,0) = A\exp\left(\frac{-x^2}{2a^2}\right)$$

This, as we will see, represents a free particle whose initial wavefunction is a Gaussian wavepacket. A is a normalization factor that can be calculated.

$$\langle \dot{X} \rangle = \frac{i}{\hbar}\langle [X,H] \rangle = \frac{i}{\hbar}\langle \underbrace{[X, \frac{P^2}{2m}]}_{\frac{i\hbar}{m}P} \rangle = \frac{1}{m}\langle P \rangle$$

$$\langle \dot{P} \rangle = \frac{i}{\hbar}\langle [P,H] \rangle = \frac{i}{\hbar}\langle \underbrace{[P, \frac{P^2}{2m}]}_{0} \rangle = 0$$

Therefore $\langle P \rangle$ is a constant. At $t=0$ we find;

$$\langle P \rangle_{t=0} = \langle \psi(0) | P | \psi(0) \rangle = A^2 \int_{-\infty}^{+\infty} dx \exp\left(-\frac{x^2}{2a^2}\right)(-i\hbar)\frac{d}{dx}\exp\left(\frac{-x^2}{2a^2}\right)$$

$$= \frac{-i\hbar A^2}{a^2}\int_{-\infty}^{+\infty} \underbrace{x\exp\left(-\frac{x^2}{a^2}\right)}_{\text{odd function}} dx = 0 \Rightarrow \boxed{\langle P \rangle = 0}$$

Therefore $\langle \dot{X} \rangle = 0$, and hence $\langle X \rangle$ is also a constant. We can easily find it at $t=0$;

$$\langle X \rangle_{t=0} = \langle \psi(0) | X | \psi(0) \rangle = A^2 \int_{-\infty}^{+\infty} dx \exp\left(-\frac{x^2}{2a^2}\right) x \exp\left(-\frac{x^2}{2a^2}\right) = A^2 \underbrace{\int_{-\infty}^{+\infty} x \exp\left(-\frac{x^2}{a^2}\right) dx}_{\text{odd function}} = 0 \Rightarrow \boxed{\langle X \rangle = 0}$$

Therefore, even though $[X, H] \neq 0$, but we have $\langle \dot{X} \rangle = 0$. The reason being that $\langle [X, H] \rangle = 0$.

Note that $[P, H] = 0$, which automatically implies that $\langle \dot{P} \rangle = 0$.

The fact that $\langle X \rangle = \langle P \rangle = 0$ for the above wavepacket can be physically understood as a result of the wavefunction symmetry under $x \to -x$.

Compatible Variables:

In classical physics one can measure two independent observables simultaneously.

An important example is the position and momentum of a particle.

This is not always the case in quantum mechanics. Consider a particle that is in the state vector $|\psi\rangle$. Consider two observables that are represented by Hermitian operators Ω and Δ.

First lets measure the observable corresponding to Ω. The probability to find value ω (one of the eigenvalues of Ω) is;

$P_{(\omega)} = |\langle\omega|\psi\rangle|^2$ where $\Omega|\omega\rangle = \omega|\omega\rangle$

The state vector collapses to $|\omega\rangle$ right after the measurement. Now we make a measurement of the observable corresponding to Δ, immediately after the first measurement.

The probability to find value λ (an eigenvalue of Δ) is given by $|\langle\lambda|\omega\rangle|^2$. Therefore:

$$P(\lambda,\omega) = |\langle\lambda|\omega\rangle|^2 \, |\langle\omega|\psi\rangle|^2$$

$P(\lambda,\omega)$ is the probability to find value ω for the first observable and then value λ for the second one.

Now we reverse the order of measurements. First measurement of the observable corresponding to Δ is made. The probability to find value λ is:

$$P(\lambda) = |\langle\lambda|\psi\rangle|^2$$

The state vector collapses to $|\lambda\rangle$, and we immediately make a measurement of the other observable. The probability to find value ω is now $|\langle\omega|\lambda\rangle|^2$. Thus:

$P(\omega,\lambda) = |\langle\omega|\lambda\rangle|^2 \, |\langle\lambda|\psi\rangle|^2$

$P(\omega,\lambda)$ is the probability to find value λ for the second observable and then value ω for the first observable.

It is clear that in general $P(\lambda,\omega) \neq P(\omega,\lambda)$

For an arbitrary state vector $|\psi\rangle$ this happens only if $|\omega\rangle = |\lambda\rangle$, for all eigenvectors $|\omega\rangle, |\lambda\rangle$ of the two operators Ω and Λ. I.E., the two operators have the same set of eigenvectors.

This will be the case only if $[\Omega, \Lambda] = 0$.

The observales corresponding to these operators are said to be compatible in this case.

Simoltaneous measurement of two observables is possible if they are compatible.

It is now clear why position and momentum of a particle cannot be determined simultaneously in quantum mechanics. They are not compatible $[X, P] = i\hbar$.

If an operator Ω commutes with the Hamiltonian H, it has the same eigenvectors as H. As we will see later on, such an operator can be used to uniquely label eigenvectors of H in the case of degeneracy.