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Identical Particles (Cont'd):

Now lets consider a realistic example of two identical particles in a three-dimensional box of volume L^3.

First consider the orbital part of the wave function. It turns out that energy eigenvalues are given by:

$$E = (n_1^2 + n_2^2 + n_3^2)\frac{\pi^2 \hbar^2}{2mL^2} \qquad n_1, n_2, n_3 = 1, 2, \ldots$$

The ground state is unique (as expected) $n_1 = n_2 = n_3 = 1$.

The first excited level has a triple degeneracy:

$$n_1 = 2,\ n_2 = n_3 = 1 \qquad n_2 = 2,\ n_1 = n_3 = 1 \qquad n_3 = 2,\ n_1 = n_2 = 1$$

And so on and so forth. We want to see whether both particles can be in the ground state $|E_0\rangle$. This depends on whether they are bosons or fermions (equivalently on their spin).

- Two identical bosons with spin 0.

In this case there is no spin part in the wavefunction. Both particles can be in the ground state:

$|\psi\rangle_{12} = |E_0 E_0\rangle$

This is by default a symmetric state. Similarly, both particles, can be in the first excited level:

$|\psi\rangle_{12} = |E_1 E_1\rangle$

Also, one can be in the ground state and another one in the first excited level. The wavefunction in this case is:

$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}\left[|E_0 E_1\rangle + |E_1 E_0\rangle\right]$

- Two identical fermions with spin-$\frac{1}{2}$.

If the two particles are in the ground state, the orbital part of the wavefunction is symmetric. However, the total wavefunction must be antisymmetric for two fermions. This implies that the spin part of the wavefunction must be antisymmetric. There is only one possibility (one spin state):

$$\frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$$

The total wavefunction is then:

$$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}|E_0 E_0\rangle \otimes \left[|+-\rangle - |-+\rangle\right]$$

Two particles can be in the ground state (or any excited level) only if one spin is up and the other is down, and for the antisymmetric combination of the spin wavefunction. This state has total spin of 0.

Now lets find the wavefunction for one particle in the ground state and the other in the first excited level. The total wavefunction must be antisymmetric. This gives us two possibilities:

(a) Orbital part symmetric, spin part antisymmetric, In this case the wavefunction will be:

$$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}\left[|E_0E_1\rangle + |E_1E_0\rangle\right] \otimes \frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right] =$$

$$\frac{1}{2}\left[|E_0E_1\rangle + |E_1E_0\rangle\right] \otimes \left[|+-\rangle - |-+\rangle\right]$$

Again the total spin of the two particle system is 0 in this case.

(b) Orbital part antisymmetric, spin part symmetric.

The wavefunction is:

$$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}\left[|E_0E_1\rangle - |E_1E_0\rangle\right] \otimes \begin{cases} |++\rangle \\ \frac{1}{\sqrt{2}}\left[|+-\rangle + |-+\rangle\right] \\ |--\rangle \end{cases}$$

There are three symmetric combinations for the spin part. This indicates that the total spin of the two particle system can assume three states, and hence the total spin is 1.

We conclude that the ground state of the two particle system has energy $2E_0$ (neglecting any interactions between the particles) and spin 0. The first excited level has energy $E_0 + E_1$ and it can have a total spin of 0 or 1.

- Three identical fermions with spin-$\frac{1}{2}$.

The total wavefunction of the three fermion system must be totally antisymmetric. As we discussed last time, there is no totally antisymmetric spin state for this system. This implies that the orbital part of the wavefunction cannot be totally symmetric.

In particular, the three identical fermions cannot be in the ground state, since the orbital part of the wavefunction $|E_0 E_0 E_0\rangle$ would be totally symmetric in that case.

The ground state of three identical fermions therefore has an energy $> 3E_0$. Let us find the wavefunction for two fermions in the ground state and one in the first excited level (total energy $2E_0 + E_1$).

Note that the orbital part of the wavefunction cannot be totally antisymmetric since two of the particles are in the same state. Therefore the spin part cannot be totally symmetric (we saw that there are four symmetric spin states in case of three fermions).

Moreover, because there are no totally antisymmetric spin states for three spin-$\frac{1}{2}$ fermions, the orbital part cannot be totally symmetric.

As a result, both the orbital and the spin part are mixed state. Now let us construct the wavefunction explicitly such that the total spin is "up", meaning two spin up $|+\rangle$ and one spin down $|-\rangle$ states involved.

let us start with the following terms and try to build a totally antisymmetric total wavefunction:

$$|E_0 E_0 E_1\rangle \otimes \left[|+-+\rangle - |-++\rangle\right]$$

Since the orbital part is symmetric under $1 \leftrightarrow 2$, the spin part must be antisymmetric under $1 \leftrightarrow 2$. We therefore have:

$$|E_0 E_0 E_1\rangle \otimes |+-+\rangle - |E_0 E_0 E_1\rangle \otimes |-++\rangle \qquad *$$

However, this is not antisymmetric under $1 \leftrightarrow 3$. Instead we get the following terms under the exchange:

$$|E_1 E_0 E_0\rangle \otimes |+-+\rangle - |E_1 E_0 E_0\rangle \otimes |++-\rangle \qquad **$$

Subtracting $**$ from $*$ results in the following wavefunction:

$$|E_0 E_0 E_1\rangle \otimes |+-+\rangle - |E_0 E_0 E_1\rangle \otimes |-++\rangle - |E_1 E_0 E_0\rangle \otimes |+-+\rangle + |E_1 E_0 E_0\rangle \otimes |++-\rangle$$

This is antisymmetric under $1 \leftrightarrow 3$, but not under $1 \leftrightarrow 2$ anymore. The trouble comes from the last two terms (the first two are antisymmetric under $1 \leftrightarrow 2$;

they are the terms we started with).

We therefore need to subtract the following terms from the above expression:

$$-|E_0 E_1 E_0\rangle \otimes |-++\rangle + |E_0 E_1 E_0\rangle \otimes |-++\rangle \qquad ***$$

Now $* - ** - ***$ results in:

normalization constant

$$|\Psi\rangle = \frac{1}{\sqrt{6}}\Big[|E_0 E_0 E_1\rangle \otimes |+-+\rangle - |E_0 E_0 E_1\rangle \otimes |-++\rangle - |E_1 E_0 E_0\rangle \otimes |+-+\rangle + |E_1 E_0 E_0\rangle \otimes |++-\rangle + |E_0 E_1 E_0\rangle \otimes |-++\rangle - |E_0 E_1 E_0\rangle \otimes |-++\rangle \Big]$$

This is the total wave function for two particles in the ground state and one in the first excited level, with the total spin "up". It can be verified that it is totally antisymmetric, i.e. $|\Psi\rangle \to -|\Psi\rangle$ under $1 \leftrightarrow 2$, $2 \leftrightarrow 3$, $1 \leftrightarrow 3$.

This example shows us how to construct totally symmetric or antisymmetric wavefunctions in general.