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Identical Particles:

Consider particles 1 and 2 that are at point r_1 and r_2, respectively, at $t=0$. The corresponding state vector in $V_1 \otimes V_2$ is:

$$|\Psi(0)\rangle_{1\otimes 2} = |r_1\rangle \otimes |r_2\rangle$$

Lets assume that the particles are distinguishabe, i.e. that some of their properties are different. This could be mass, electric charge, spin, etc; any quantum number that labels a particle.

Now the system evolves and we make a measurement at a later time $t=T$. We find that one of the particles is at point r_3 and the other one is at point r_4 at $t=T$. Depending on which particle is at r_3, there are two distinct possibilities:

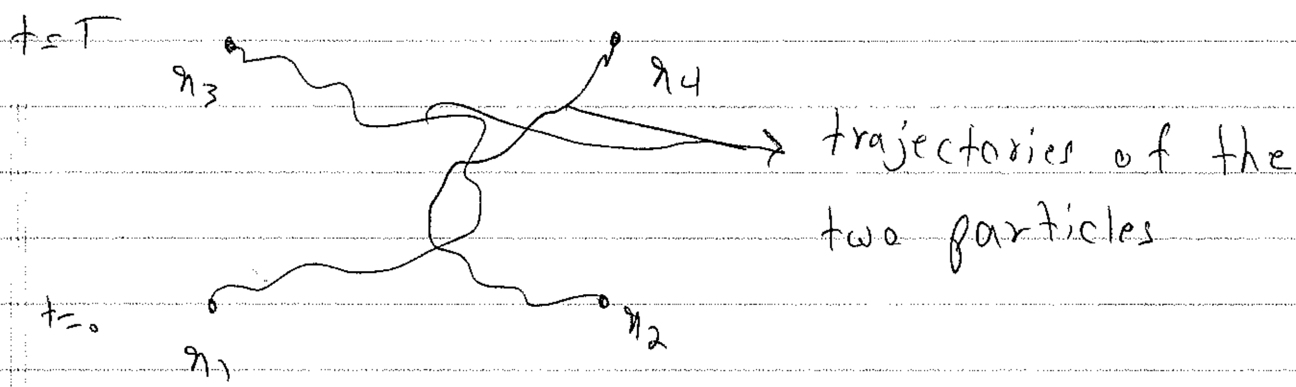
$$|\Psi(T)\rangle_{1\otimes 2} = |x_3\rangle \otimes |x_4\rangle \quad \text{(Particle 1 at } x_3\text{)}$$

$$|\Psi(T)\rangle_{1\otimes 2} = |x_4\rangle \otimes |x_3\rangle \quad \text{(Particle 2 at } x_3\text{)}$$

Now lets assume that the two particles are identical, i.e. that they have the same mass, charge, spin, etc. Then there will be no quantum number under which the two particles are labeled differently.

Now the question is whether we can say the particle that is found at x_3 at $t=T$ started at x_1 or x_2 at $t=0$.

The answer to this question is positive in classical mechanics. Since position and momentum can be measured simultaneously, each particle has a well defined trajectory. Thus the particle that is at x_3 at $t=T$ can be traced back to its origin at $t=0$ (along its trajectory). This way we can say which of the two particles is at r_3 and which one is at r_4 at $t=T$.



In quantum mechanics, however, there is no such a thing as a well defined trajectory. This implies that there is no way to trace back the particle at r_3 or r_4 and find where it was at $t=0$. Therefore, all we can say is that one of the particles is at r_3 and the other one is at r_4.

In other words, the distinction between 1 and 2 is all gone. As a result, the state vector at $t=T$ is a superposition of $|r_3\rangle \otimes |r_4\rangle$ (particle 1 at r_3, particle 2 at r_4) and $|r_4\rangle \otimes |r_3\rangle$ (particle 1 at r_4, particle 2 at r_3):

$$|\Psi(T)\rangle_{1\otimes 2} = a|r_3\rangle \otimes |r_4\rangle + b\,|r_4\rangle \otimes |r_3\rangle$$

An important thing to note is that swapping of the two particles (equivalent to $r_3 \leftrightarrow r_4$ exchange) does not change anything physically because the particles are identical. If physics remains the same, so will the probability distribution. This leads to:

$$\left|\langle r_4| \otimes \langle r_3|\,\Psi(T)\rangle_{1\otimes 2}\right|^2 = \left|\langle r_4| \otimes \langle r_3|\,\Psi'(T)\rangle_{1\otimes 2}\right|^2$$

Where:

$$|\Psi'(T)\rangle_{1\otimes 2} = a|r_4\rangle \otimes |r_3\rangle + b\,|r_3\rangle \otimes |r_4\rangle$$

Thus:

$$|a|^2 = |b|^2 \Rightarrow a = b\,e^{i\phi}$$

It turns out from spin-statistics theorem that in three dimensions two possibilities exist $a=b$ or $a=-b$. The wavefunction of the two identical particle system is symmetric in the first case and is antisymmetric in the second case.

Particles for which $a=b$ are called bosons and have integer spins (for example, photon). Particles for which $a=-b$ are called fermions and their spin is an odd multiple of $\frac{1}{2}$ (for example, electron).

Side Note: This is true in three dimensions. In two spatial dimensions actually all phase $0 \leq \phi < 2\pi$ are allowed. This leads to the notion of "anyons". They have important applications in the fractional quantum Hall effect and superconductivity. Here we focus on three dimensions only.

Let us summarize the important points about identical particles:

1- The total wavefunction for two identical bosons is symmetric under swapping the particles.

2- The total wave function for two identical fermions is antisymmetric under swapping the particles.

By total wave function we mean that all possible degrees of freedom are taken into account (orbital and spin degrees of freedom):

$$|\psi\rangle_{tot} = |\psi\rangle_{orbit} \otimes |\psi\rangle_{spin}$$

3- For N identical particles, the total wave function is symmetric (antisymmetric) under swapping every pair of particles if they are bosons (fermions).

This come in a straightforward manner from that for two particles (for details see Shankar).

A very important consequence of 2 is that two identical fermions cannot exist in the same quantum state.

Consider two identical particle one in state $|1\rangle$ and the other in state $|2\rangle$. The wavefunction for the two particle system is:

$$|\psi\rangle_{1\otimes 2} = \underbrace{\frac{1}{\sqrt{2}}}_{\text{normalization constant}} \left[|1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle \right] \qquad \text{(bosons)}$$

$$|\psi\rangle_{1\otimes 2} = \frac{1}{\sqrt{2}} \left[|1\rangle \otimes |2\rangle - |2\rangle \otimes |1\rangle \right] \qquad \text{(fermions)}$$

If the two states were the same, then $|\psi\rangle_{1\otimes 2} = 0$ in the fermionic case. This is the so called Pauli enclusion principle. As one consequence, this tell us why electrons feel different orbits in an atom instead of all being in the ground state.

Note, however, that bosons can all be in the same quantum state. Actually, this is the reason why quantum fields with integer spin have a classical limit (like the electromagnetic field).

We will discuss consequences of identical particles in quantum mechanics in more detail later on.