Lec 30: 11/06/2009

More Particles in One Dimension:

So far we have considered one particle systems in one dimension. Systems with multiple degrees of freedom have more particles in one or more dimensions.

We start with more particles in one dimension. For simplicity, consider two particles in one dimension. The classical Hamiltonian consists of kinetic energy terms for the particles and the potential energy term:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(x_1, x_2)$$

The energy eigenvalues of the system are found by solving the following equation:

$$-\frac{\hbar^2}{2m_1}\frac{\partial^2 \psi(x_1, x_2)}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2 \psi(x_1, x_2)}{\partial x_2^2} + V(x_1, x_2)\psi(x_1, x_2) = E\,\psi(x_1, x_2)$$

In general, it is very difficult or impossible to solve this equation. However, great simplification is made in two cases:

1- Non-interacting particles in an external potential: In this case $V(r_1, r_2) = V_1(r_1) + V_2(r_2)$. Since both kinetic energy and potential energy terms depend on r_1 or r_2 only, we can use separation of variables:

$$\psi(r_1, r_2) = \psi_1(r_1)\,\psi_2(r_2)$$

Two ordinary differential equations are then found:

$$-\frac{\hbar^2}{2m_1}\frac{d^2\psi_1(r_1)}{dr_1^2} + V_1(r_1)\,\psi_1(r_1) = E_1\,\psi_1(r_1)$$

$$-\frac{\hbar^2}{2m_2}\frac{d^2\psi_2(r_2)}{dr_2^2} + V_2(r_2)\,\psi_2(r_2) = E_2\,\psi_2(r_2)$$

The energy eigenvalues of the two particle system are simply the sum of energy eigenvalues in these two equations:

$$E = E_1 + E_2$$

Because the particles do not interact with each other, the two particle system is a union of two independent one particle systems.

2- Interacting particles with no external potential:

In this case, assuming translational invariance, the potential depends on $x_1 - x_2$ only. A convenient coordinate system to use is that of the center-of-mass and relative position of the two particles:

$$X_{CM} \equiv \frac{m_1 X_1 + m_2 X_2}{m_1 + m_2}$$

$$X \equiv X_1 - X_2$$

Defining the conjugate momenta:

$$P_{CM} = -i\hbar \frac{\partial}{\partial X_{CM}} , \quad p = -i\hbar \frac{\partial}{\partial x}$$

The commutation relations $[X_{CM}, P_{CM}] = [x, p] = i\hbar$ directly follow from the commutation relations

$[X_1, P_1] = [X_2, P_2] = i\hbar$.

Moreover, it turns out that:

$$-\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial X_{CM}^2} - \frac{\hbar^2}{2\nu}\frac{\partial^2}{\partial x^2}$$

Here $M = m_1 + m_2$ is the total mass and $\nu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of a two-body system.

Now we can use separation of variables and write:

$$\Psi(x_1, x_2) = \Psi_{CM}(X_{CM})\,\Psi_{rel}(x)$$

Then we find:

$$-\frac{\hbar^2}{2M}\frac{d^2\Psi_{CM}}{dX_{CM}^2} = E_{CM}\Psi_{CM}$$

$$-\frac{\hbar^2}{2\nu}\frac{d^2\Psi_{rel}}{dx^2} + V(x)\Psi_{rel} = E_{rel}\Psi_{rel}$$

The first equation is just that for a free particle, thus:

$$\Psi_{CM} = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\pm \frac{i P_{CM}}{\hbar} X_{CM}} \qquad E_{CM} = \frac{P_{CM}^2}{2M}$$

$$E = E_{CM} + E_{rel}$$

Jacobi Coordinates:

The coordinates X_{CM}, x are called Jacobi coordinates.

In general for N particles we have:

$$H = \sum_{i=1}^{N} \frac{\hbar^2}{2m_i} \frac{\partial^2}{\partial x_i^2} + V(x_1, \dots, x_N)$$

Jacobi coordinates are then defined as:

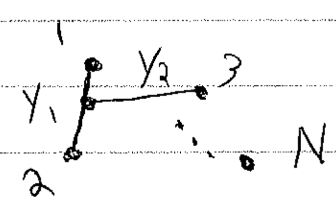
$$Y_1 = x_1 - x_2$$

$$Y_2 = x_3 - \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$\vdots$$

$$X_N = \frac{m_1 x_1 + m_2 x_2 + \dots + m_N x_N}{m_1 + m_2 + \dots + m_N}$$

Y_1 is the relative position of particles 1 and 2, Y_2 is the position of particle 3 relative to the center of mass of 1 and 2, etc. Finally, X_N is the center of mass of the whole N body system.



Again, one can check that:

$$[Y_1, P_1^Y] = \dots = [Y_N, P_N^Y] = i\hbar$$

$$P_1^Y = -i\hbar \frac{\partial}{\partial Y_1}, \dots, P_N = -i\hbar \frac{\partial}{\partial Y_N}$$

And:

$$\sum_{i=1}^{N} -\frac{\hbar^2}{2m_i} \frac{\partial^2}{\partial x_i^2} = \sum_{i=1}^{N} -\frac{\hbar^2}{2\nu_i} \frac{\partial^2}{\partial Y_i^2}$$

Where:

$$\frac{1}{\nu_1} = \frac{1}{m_1} + \frac{1}{m_2}, \quad \frac{1}{\nu_2} = \frac{1}{\nu_1} + \frac{1}{m_3}, \dots, \nu_N = m_1 + m_2 + \dots + m_N$$

Note that Y_N is the center of mass of the whole N body system X_{CM} and ν_N is the total mass of the system M. If the particles interact with each other but there is no external potential, then $V(x_1, \dots, x_N) = V(Y_1, \dots, Y_{N-1})$. Hence:

$$-\frac{\hbar^2}{2M} \frac{\partial^2 \psi}{\partial X_{CM}^2} + \sum_{i=1}^{N-1} -\frac{\hbar^2}{2\nu_i} \frac{\partial^2 \psi}{\partial Y_i^2} + V(Y_1, \dots, Y_{N-1})\psi = E\psi$$

Again, we can use separation of variables:

$$\psi(x_1, \ldots, x_N) = \psi_{CM}(X_{CM})\, \psi_{rel}(y_1, \ldots, y_{N-1})$$

And:

$$-\frac{\hbar^2}{2M} \frac{\partial^2 \psi_{CM}}{\partial X_{CM}^2} = E_{CM}\, \psi_{CM}$$

$$\sum_{i=1}^{N-1} -\frac{\hbar^2}{2\mu_i} \frac{\partial^2 \psi_{rel}}{\partial y_i^2} + V(y_1, \ldots, y_{N-1})\, \psi_{rel} = E_{rel}\, \psi_{rel}$$

The first equation is that of a free particle, implying that:

$$\psi_{CM}(X_{CM}) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\pm \frac{i p_{CM}}{\hbar} X_{CM}} \qquad E_{CM} = \frac{p_{CM}^2}{2M}$$

$$E = E_{CM} + E_{rel}$$

ψ_{rel} and E_{rel} depend on the exact form of $V(y_1, \ldots, y_{N-1})$.