Lec 21: 10/14/2009

Path Integral in Quantum Mechanics (Cont'd):

Let us find $U(x,t;x',0)$ for a free particle by computing S_{cl}. As we mentioned, the classical path is all we need in this case. The classical trajectory obeys the Euler-Lagrange equation:

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad L = \frac{1}{2} m\dot{x}^2$$

Thus:

$$m\ddot{x}_{cl} = 0 \Rightarrow x_{cl}(\tau) = a + b\tau$$

$$x_{cl}(0) = x' \ , \ x_{cl}(t) = x \Rightarrow x_{cl}(\tau) = x' + \frac{x - x'}{t}\tau$$

Then:

$$L = \frac{1}{2}m\dot{x}^2 = \frac{1}{2} m \frac{(x-x')^2}{t^2} \Rightarrow S_{cl} = \frac{1}{2}\frac{m(x-x')^2}{t}$$

And:

$$U(x,t;x',0) = A\, e^{\frac{iS_{cl}}{\hbar}} = A\, e^{\frac{im(x-x')^2}{2\hbar t}}$$

Now we should try and find A. Note that this is a time-dependent constant. In the limit $t \to 0$ we have;

$$\lim_{t\to 0} U(x,t;x',0) = \lim_{t\to 0} \langle x| e^{-\frac{iHt}{\hbar}} |x'\rangle = \delta(x-x')$$

Thus;

$$\lim_{t\to 0} \left[A e^{\frac{im(x-x')^2}{2\hbar t}} \right] = \delta(x-x')$$

Defining $\Delta^2 = \frac{2i\hbar t}{m}$, we have;

$$U(x,t;x',0) = A \exp\left[\frac{-(x-x')^2}{\Delta^2}\right] \Rightarrow A = \frac{1}{\pi \Delta^2}$$

This results in;

$$A = \left(\frac{m}{2i\pi\hbar t}\right)^{\frac{1}{2}}$$

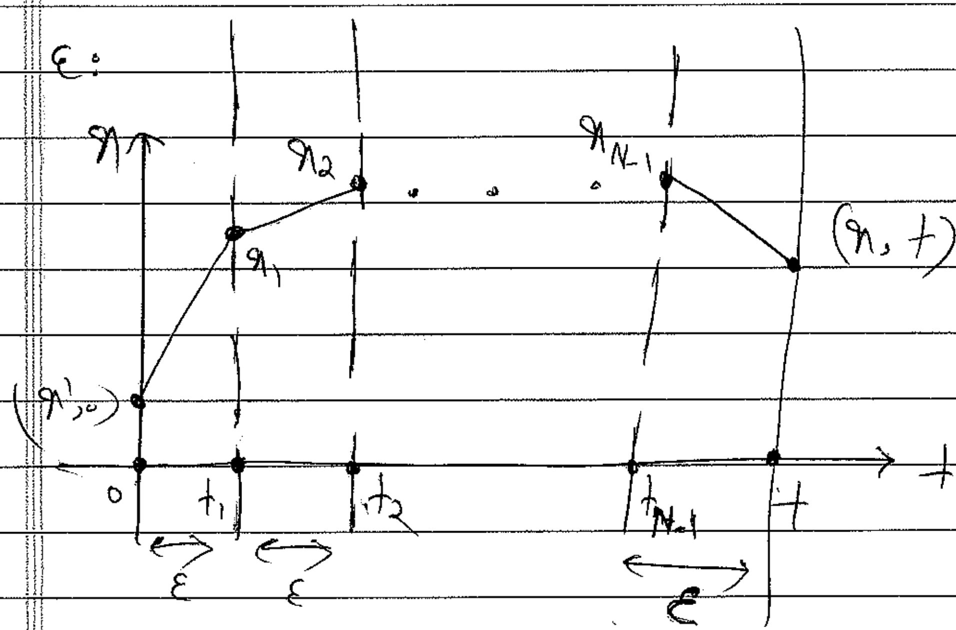
Note however that there can still be another piece $f(t)$ such that $U(x,t;x',0) = f(t) \left(\frac{m}{2i\pi\hbar t}\right)^{\frac{1}{2}} e^{\frac{im(x-x')^2}{2\hbar t}}$

and $\lim_{t\to 0} f(t) \to 1$. This is the case in general. But for a free particle we actually have $f = 1$.

The reason being that we cannot construct a nontrivial dimensionless combination of the parameters involved.

a time-dependent [illegible]

For a free particle the classical path is a straight line between the point $(x',0)$ and (x,t). Now that we have computed S_{cl} for a free particle, we can actually perform the sum over all paths to explicitly verify that $U(x,t;x',0) \sim e^{\frac{iS_{cl}}{\hbar}}$ for a free particle. Note that an arbitrary path between the two points can be approximated by a linewise path consisting of line pieces in time steps of length ε:



$$\varepsilon = \frac{t}{N}$$

Where we will take the limit $\epsilon \to 0$ $(N \to \infty)$ at the end.

For a given path we have:

$$S = \int_0^t \frac{1}{2} m \dot{x}^2 \, d\tau = \int_0^{\epsilon} \frac{1}{2} m \dot{x}^2 \, d\tau + \int_{\epsilon}^{2\epsilon} \frac{1}{2} m \dot{x}^2 \, d\tau + \cdots + \int_{(N-1)\epsilon}^{N\epsilon} \frac{1}{2} m \dot{x}^2 \, d\tau \simeq \frac{1}{2} m \frac{(x_1 - x')^2}{\epsilon} + \frac{1}{2} \frac{m (x_2 - x_1)^2}{\epsilon} + \cdots + \frac{1}{2} \frac{m (x - x_{N-1})^2}{\epsilon}$$

We can construct all possible paths by changing $x_1, \ldots, x_{N-1}$ independently from $-\infty$ to $+\infty$ (note that x at the end points is fixed). Therefore:

$$\sum_{\text{all paths}} e^{iS/\hbar} = \int_{-\infty}^{+\infty} dx_{N-1} \cdots \int_{-\infty}^{+\infty} dx_2 \int_{-\infty}^{+\infty} dx_1 \; e^{iS/\hbar}$$

Where:

$$e^{iS/\hbar} = \exp\left[\frac{im (x_1 - x')^2}{2\epsilon} + \frac{im (x_2 - x_1)^2}{2\epsilon} + \cdots + \frac{im (x - x_{N-1})^2}{2\hbar} \right]$$

First lets perform integration over x_1. The first two terms in the brackets are relevant in this

Case:

$$\int_{-\infty}^{+\infty} dx_1 \exp\left[\frac{im(x_1 - x')^2}{2\varepsilon} + \frac{im(x_2 - x_1)^2}{2\varepsilon}\right] =$$

$$\int_{-\infty}^{+\infty} \exp\left\{\frac{im}{2\varepsilon}\left[2x_1^2 - x_1(x' + x_2) + x_2^2 + x'^2\right]\right\}$$

$$2x_1^2 - x_1(x' + x_2) + x'^2 + x_2^2 = 2\left[x_1^2 - x_1(x' + x_2) + \frac{1}{2}(x'^2 + x_2^2)\right] = 2\left\{\left[x_1 - (x_2 + x')\right]^2 + \frac{1}{2}(x'^2 + x_2^2) - \frac{1}{4}(x_2 + x')^2\right\} = 2\left\{\left[x_1 - (x_2 + x')\right]^2 + \frac{1}{4}(x' + x_2)^2\right\}$$

Thus:

$$\int_{-\infty}^{+\infty} dx_1 \exp\left\{\frac{im}{2\varepsilon}\left[(x_1 - x')^2 + (x_2 - x_1)^2\right]\right\} =$$

$$\underbrace{\int_{-\infty}^{+\infty} dx_1 \exp\left\{\frac{im}{2\varepsilon}\left[x_1 - (x_2 + x')\right]^2\right\}}_{\text{Const.}} \times \exp\left[\frac{im(x_2 - x')^2}{4\varepsilon}\right]$$

Implying that:

$$\sum_{\substack{\text{all}\\ \text{paths}}} e^{iS/\hbar} (\text{Const}) \int_{-\infty}^{+\infty} dx_{N-1} \cdots \int_{-\infty}^{+\infty} dx_2 \exp\left[\frac{im(x_2 - x')^2}{4\varepsilon} + \frac{im(x_3 - x_2)^2}{2\varepsilon} + \cdots + \frac{im(x - x_{N-1})^2}{2\varepsilon}\right]$$

Now only the first two terms in the brackets are relevant as far as integration over x_2 is concerned. This is similar to what we did when taking the integral over x_1. At each step we have a Gaussian integral, which yields a constant, multiplied by an exponential factor. Integrating over $x_1, \ldots, x_{N-1}$ we find:

$$\sum_{\text{all paths}} e^{iS/\hbar} = (\text{Const.})\, e^{\frac{im(x-x')^2}{2\hbar N\varepsilon}}$$

But note that $N\varepsilon = t$. Eventually, when we take the limit $\varepsilon \to 0$ $(N \to \infty)$, we find:

$$U(x,t;x',0) = \lim_{N\to\infty}\left[(\text{Const.})\, e^{\frac{im(x-x')^2}{2\hbar t}}\right] =$$

$$A\, e^{\frac{im(x-x')^2}{2\hbar t}} \qquad A = \lim_{N\to\infty}(\text{Const.})$$

This is exactly what we stated before: only the classical path matters to find $U(x,t;x',0)$ for a free particle.