Lect 2: 08/26/2009

Linear Operators:

An operator Ω transforms any vector $|V\rangle$ to another vector $|V'\rangle$ in the vector space:

$$\Omega|V\rangle = |V'\rangle$$

Ω is linear if:

$$\Omega(\alpha|V_i\rangle) = \alpha\,\Omega|V_i\rangle$$

$$\Omega\left(\alpha|V_i\rangle + \beta|V_j\rangle\right) = \alpha\,\Omega|V_i\rangle + \beta\,\Omega|V_j\rangle$$

The simplest linear operator is the identity operator I:

$$I|V\rangle = |V\rangle \quad \forall\, |V\rangle \in \mathbb{V}$$

Another example is rotation about a given axis.

Commutator of two operators Ω, Λ is defined as:

$$[\Omega, \Lambda] \equiv \Omega\Lambda - \Lambda\Omega$$

The inverse of Ω, denoted by Ω^{-1}, is defined as:

$$\Omega\Omega^{-1} = \Omega^{-1}\Omega = I$$

Note:

$$I^{-1} = I$$

The inverse of rotation operator is a rotation about the same axis in the opposite direction.

Consider an orthonormal basis $\{|1\rangle, \ldots, |n\rangle\}$ that spans an n dimensional vector space $\mathbb{V}^n$.

Any vector $|V\rangle$ can be written as:

$$|V\rangle = \sum_{i=1}^{n} v_i \, |i\rangle$$

The identity operator is given by:

$$I = \sum_{i=1}^{n} |i\rangle\langle i|$$

The matrix representation of an operator Ω in this basis is given by:

$$\Omega_{ij} \equiv \langle i | \Omega | j \rangle$$

For the identity operator we have:

$$I_{ij} = \delta_{ij} \qquad (I \text{ is } n \text{ dimensional unit matrix})$$

Knowing Ω_{ij}, we can find $\Omega |V\rangle$ for any $|V\rangle$:

$$|V\rangle = \sum_j v_j |j\rangle \Rightarrow \overbrace{\Omega|V\rangle}^{|V'\rangle =} = \sum_j v_j \Omega |j\rangle$$

$$\langle i | V' \rangle = \sum_j \langle i | \Omega | j \rangle v_j = \sum_j v_j \Omega_{ij} = v'_j$$

Example: Consider rotation by angle θ about the z axis. The matrix representation of this operator is:

$$\Omega \leftrightarrow \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The adjoint of operator Ω, denoted by $\Omega^\dagger$, is defined as:

$\langle i|\Omega^\dagger|j\rangle = \langle j|\Omega|i\rangle^*$

In other words $\Omega^\dagger_{ij} = \Omega^*_{ji}$. The elements of matrix representation of $\Omega^\dagger$ are obtained from those of Ω by taking the transpose and complex conjugate of matrix representation of Ω.

For products of two operators:

$(\Omega\Lambda)^\dagger = \Lambda^\dagger \Omega^\dagger$

An operator Ω is Hermitian if $\Omega^\dagger = \Omega$.

An operator U is unitary if $UU^\dagger = I \; (U^\dagger = U^{-1})$.

The Eigenvalue Problems

If ;

$\Omega|V\rangle = \omega|V\rangle$

$|V\rangle$ is called an eigenket of Ω with eigenvalue ω.

Hermitian operators have real eigenvalues:

$\langle V | \Omega^{+} | V \rangle = \langle V | \Omega | V \rangle^{*}$

$\Omega = \Omega^{+} \Rightarrow \langle V | \Omega | V \rangle = \langle V | \Omega | V \rangle^{*}$

$\langle V | \Omega | V \rangle = \langle V | V \rangle \omega \quad , \langle V | \Omega | V \rangle^{*} = \langle V | V \rangle^{*} \omega^{*}$

$= \langle V | V \rangle \omega^{*}$

Thus $\omega = \omega^{*}$. Moreover, the eigenkets corresponding to unequal eigenvalues are orthogonal. Lets assume:

$\Omega | V_1 \rangle = \omega_1 | V_1 \rangle \qquad \Omega | V_2 \rangle = \omega_2 | V_2 \rangle \quad \omega_1 \neq \omega_2$

$\langle V_1 | \Omega^{+} | V_2 \rangle = \langle V_2 | \Omega | V_1 \rangle^{*}$

$\langle V_1 | \Omega^{+} | V_2 \rangle = \langle V_1 | \Omega | V_2 \rangle = \omega_2 \langle V_1 | V_2 \rangle$

$\langle V_2 | \Omega | V_1 \rangle^{*} = \omega_1^{*} \langle V_2 | V_1 \rangle^{*} = \omega_1 \langle V_1 | V_2 \rangle$

Since $\omega_1 \neq \omega_2$, we must have $\langle V_1 | V_2 \rangle = 0$ in order for $\langle V_1 | \Omega^{+} | V_2 \rangle = \langle V_2 | \Omega | V_1 \rangle^{*}$.

Eigenvalues of unitary operators are complex numbers of modulus one:

$$U\,|v\rangle = u\,|v\rangle \Rightarrow \left(U\,|v\rangle\right)^{\dagger} = \langle v|\,U^{\dagger} = \langle v|\,u^{*}$$

But:

$$\langle v|\underbrace{U^{\dagger}U}_{I}|v\rangle = \langle v|\,u^{*}u\,|v\rangle \Rightarrow u^{*}u = 1$$

Again, eigenkets corresponding to unequal eigenvalues are orthogonal:

$$U\,|v_1\rangle = u_1\,|v_1\rangle \qquad U\,|v_2\rangle = u_2\,|v_2\rangle \qquad u_1 \neq u_2$$

$$\langle v_2|\underbrace{U^{\dagger}U}_{I}|v_1\rangle = v_2^{*}\,v_1\,\langle v_2|v_1\rangle = \langle v_2|v_1\rangle$$

But $v_1 \neq v_2$, and hence $v_1 v_2^{*} \neq 1$. This requires that $\langle v_1|v_2\rangle = 0$.

In order to find eigenvalues of an operator, we need to solve the characteristic