Lec 16: 10/02/2009

## Harmonic Oscillator in the Energy Basis (Cont'd):

Note that the Hamiltonian is positive definite, implying that the lowest energy eigenvalue is positive. Therefore, there must be a state $|0\rangle$ such that $a|0\rangle = 0$:

$$a|0\rangle = 0 \Rightarrow H|0\rangle = \frac{\hbar\omega}{2}|0\rangle$$

This is the ground state. Other energy eigenstates are obtained by acting $a^\dagger$ on $|0\rangle$:

$$a^\dagger|0\rangle = C_1|1\rangle \quad , \quad a^\dagger|1\rangle = C_2|2\rangle , \ldots$$

$$H|n\rangle = (n+\tfrac{1}{2})\hbar\omega \qquad a^\dagger a|n\rangle = n|n\rangle$$

We can now find $C_\varepsilon, C_{\varepsilon+1}$. Note that:

$$a|n\rangle = C_n|n-1\rangle \Rightarrow \langle n|a^\dagger a|n\rangle = |C_n|^2 \underbrace{\langle n-1|n-1\rangle}_{1}$$

$$= |C_n|^2 = n \Rightarrow C_n = \sqrt{n} \quad \text{(up to a phase)}$$

Similarly, $C_{n+1} = \sqrt{n+1}$. Therefore:

$a|n\rangle = \sqrt{n}\,|n-1\rangle$ , $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

The operator $a^\dagger a$ is called the number operator $N$.

Using lowering and raising operators, we can calculate the expectation value and uncertainty in position and momentum easily. For example:

$$\langle n|X|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}} \langle n|(a+a^\dagger)|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}$$

$$\Big[\underbrace{\langle n|n-1\rangle}_{0}\sqrt{n} + \underbrace{\langle n|n+1\rangle}_{0}\sqrt{n+1}\Big] = 0$$

$$\langle n|P|n\rangle = i\left(\frac{\hbar m\omega}{2}\right)^{\frac{1}{2}} \langle n|(a^\dagger - a)|n\rangle = i\left(\frac{\hbar m\omega}{2}\right)^{\frac{1}{2}}$$

$$\Big[\underbrace{\langle n|n+1\rangle}_{0}\sqrt{n+1} - \underbrace{\langle n|n-1\rangle}_{0}\sqrt{n}\Big] = 0$$

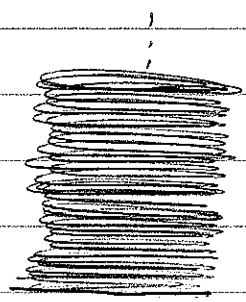
Let us briefly discuss two important consequences of the fact that the energy spectrum of quantum harmonic oscillator is discrete.

Specific Heat of Crystals:

Specific heat C is defined as:

$\Delta Q = C \Delta T$

Where Q is the amount of heat absorbed by an object. The atoms of a crystal behave as harmonic oscillators. Classically, the energy spectrum is continuous:



$E = 0$

Now if the crystal is in contact with a heat reservoir at temperature T, the oscillators get excited. Due to Maxwell-Boltzmann distribution, the amount of energy E that can be received from the reservoir is exponentially suppressed

$\propto e^{-\frac{\Delta E}{kT}}$.

For a classical oscillator, since the spectrum is continuous, no matter how small $T$ is an energy exchange $E \sim kT$ can happen and take the oscillator from ground state $E=0$ to an excited state. One therefore expects the heat capacity $C$ to be a nonzero constant as $T \to 0$.

However, it is observed that $C \to 0$ as $T \to 0$.

This can be explained by quantum harmonic oscillator. In this case the spectrum is discrete:

$\vdots$

$\frac{7}{2}\hbar\omega$

$\frac{5}{2}\hbar\omega$

$\frac{3}{2}\hbar\omega$

$E = \frac{1}{2}\hbar\omega$

Therefore an oscillator in the ground state needs a minimum $\hbar\omega$ of energy to get excited. An energy exchange in the amount of $\hbar\omega$ is exponentially suppressed as $T \to 0$ ($kT \ll \hbar\omega$). Therefore as $T \to 0$, we find that $C \to 0$.

## Black-body Radiation:

Consider a cavity with coated walls. In finite temperature, we get standing electromagnetic waves at all frequencies $\omega = ck$ ($k$ the wave number) that fill the cavity.

Clasically, the energy of electromagnetic waves does not depend on the frequency, but rather on the amplitude. Therefore it takes the same amount of energy to excite all modes with $0 < \omega < \infty$. The number of modes whose frequency is between $\omega$ and $\omega + d\omega$ goes proportional to

$\omega^2 d\omega$ (in three dimensions). Therefore, the amount of energy between wavelegths $\lambda$ and $\lambda + d\lambda$ (the spectral density) follows:

$$I(\lambda, T)\, d\lambda \propto \lambda^{-4}\, d\lambda \qquad \left(d\omega \propto \frac{d\lambda}{\lambda^2}\right)$$

This give rise to a divergent behavior $\propto \omega^4$ at high frequencies (Raileigh - Jeans law).

Again this is not what we observe experimentally. Instead $I$ is given by the Planck's law wher $I \propto e^{-\frac{\hbar\omega}{kT}}$ when $\hbar\omega \gg kT$.

This can be understood from the discrete nature of the spectrum. Quantum mechanically, an electromagnetic wave of frequency $\omega$ is nothing but a harmonic oscillator with that frequency. The energy eigenvalues depend on $\omega$ unlike the classical case.

As a result, modes with frequency $\hbar\omega \gg kT$ are exponentially suppressed. This give rise to finiteness of the total energy in the cavity and cures the classical divergence.