Lec 13: 09/23/2009

The Continuity Equation for Probability:

The probability density $|\Psi(x,t)|^2$ can vary as a function of time and position. However, the total probability $\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = 1$ is conserved. This is easily seen because of the unitary evolution of the state vector:

$$\langle \Psi(t) | \Psi(t) \rangle = \langle \Psi(0) | \underbrace{U^\dagger U}_{I} | \Psi(0) \rangle = \langle \Psi(0) | \Psi(0) \rangle = 1$$

Conservation of the total probability suggests that there exists a continuity equation, similar to that for electric current-electric charge (because of charge conservation) or Poynting vector-energy density (because of electromagnetic energy conservation) in electromagnetism.

To show this, consider Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V\psi=i\hbar\frac{\partial\psi}{\partial t}$$

Taking the complex conjugate of both sides, we find:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2}+V\psi^*=-i\hbar\frac{\partial\psi^*}{\partial t} \quad (V \text{ is real})$$

Multiplying the first equation by ψ^* and the second one by ψ, and then subtracting, we get:

$$-\frac{\hbar^2}{2m}\left(\psi^*\frac{\partial^2\psi}{\partial x^2}-\psi\frac{\partial^2\psi^*}{\partial x^2}\right)=i\hbar\frac{\partial(\psi^*\psi)}{\partial t} \Rightarrow$$

$$\frac{\partial j(x,t)}{\partial x}+\frac{\partial \rho(x,t)}{\partial t}=0$$

Where:

$$\rho(x,t)=|\psi(x,t)|^2 \quad , \quad j(x,t)=\frac{\hbar}{2im}\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right)$$

In three dimensions we have:

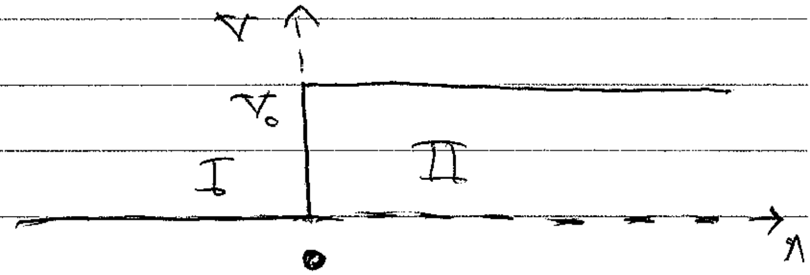
$$\rho(x,y,z,t)=|\psi(x,y,z,t)|^2 \quad , \quad \vec{j}(x,y,z,t)=\frac{\hbar}{2im}\left(\psi^*\vec{\nabla}\psi-\psi\vec{\nabla}\psi^*\right)$$

This can be used in scattering problems.

Step Potential:

Consider a potential that is the step function:

$$\begin{cases} V(x) = 0 & x < 0 \\ V(x) = V_0 & x > 0 \end{cases}$$



First of all, note that this system has no bound states: for $E < V_0$ the solution to the eigenvalue problem is oscillatory in region I (thus does not vanish at $-\infty$), and for $E > V_0$ the solution is oscillatory everywhere (hence nonvanishing at $\pm\infty$)

For $E < V_0$, we have (up to a normalization constant)

$$\begin{cases} \psi_I(x) = e^{ikx} + A e^{-ikx} & k = \sqrt{\dfrac{2mE}{\hbar^2}} \\ \psi_{II}(x) = B e^{-\kappa x} & \kappa = \sqrt{\dfrac{2m(V_0 - E)}{\hbar^2}} \end{cases}$$

For $E > V_0$, we get:

$$\begin{cases} \psi_I(x) = e^{ik_1 x} + Ae^{-ik_1 x} & k_1 = \sqrt{\dfrac{2mE}{\hbar^2}} \\ \psi_{II}(x) = Be^{ik_2 x} + Ce^{-ik_2 x} & k_2 = \sqrt{\dfrac{2m(E-V_0)}{\hbar^2}} \end{cases}$$

Note that V is finite everywhere. Therefore ψ and $\frac{d\psi}{dx}$ must be continuous. This gives two equations at $x=0$.

For any value of E that we choose, we get two equations and two unknowns if $E < V_0$, and two equations and three unknowns if $E > V_0$. Thus, we find solutions to the eigenvalue problem for $\forall E > 0$.

Now lets consider the following scattering problem. A particle coming from $-\infty$ scatters off the step potential. What is the solution to the eigenvalue problem in this case?

Note that for classical particle there will be no scattering. If $E > V_0$, the particle will climb the barrier and move to region II.

The solution to this scattering problem is:

$$\begin{cases} \psi_{I}(x) = \underbrace{e^{ik_1 x}}_{\psi_I} + \underbrace{A e^{-ik_1 x}}_{\psi_R} & k_1 = \sqrt{\dfrac{2mE}{\hbar^2}} \\ \psi_{II}(x) = \underbrace{B e^{ik_2 x}}_{\psi_T} & k_2 = \sqrt{\dfrac{2m(E-V_0)}{\hbar^2}} \end{cases}$$

Similar to electromagnetism, we have an incident wave that is partly reflected (A) and partly transmitted (B).

Using the continuity of ψ and $\frac{d\psi}{dx}$ at $x=0$, we find:

$$A = \frac{k_1 - k_2}{k_1 + k_2}, \quad B = \frac{2k_1}{k_1 + k_2}$$

In analogy to electromagnetism, the reflection and transmission coefficients R, T are defined as:

$$R = \frac{j_R}{j_I} \;, \quad T = \frac{j_T}{j_I}$$

Where;

$$j_I = \frac{\hbar}{2im}\left(\psi_I^* \frac{d\psi_I}{dx} - \psi_I \frac{d\psi_I^*}{dx}\right) = \frac{\hbar k_1}{m}$$

$$j_R = \frac{\hbar}{2im}\left(\psi_R^* \frac{d\psi_R}{dx} - \psi_R \frac{d\psi_R^*}{dx}\right) = \frac{\hbar k_1}{m} |A|^2$$

$$j_T = \frac{\hbar}{2im}\left(\psi_T^* \frac{d\psi_T}{dx} - \psi_T \frac{d\psi_T^*}{dx}\right) = \frac{\hbar k_2}{m} |B|^2$$

It is easy to see that:

$$R + T = 1$$

As we expect.

We can also solve the scattering problem for $E < V_0$

Again, note that the classical particle will be reflected at $x = 0$ and cannot climb the barrier.

The solution in this case is;

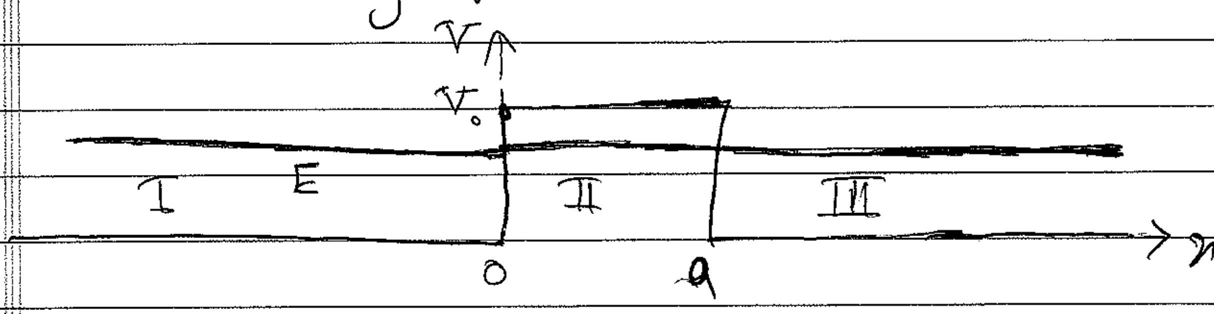
$$\begin{cases} \psi_I(x) = \underbrace{e^{ikx}}_{\psi_I} + \underbrace{A e^{-ikx}}_{\psi_R} & k = \sqrt{\dfrac{2mE}{\hbar^2}} \\ \psi_{II}(x) = \underbrace{B e^{-kx}}_{\psi_T} & k = \sqrt{\dfrac{2m(-E+V_0)}{\hbar^2}} \end{cases}$$

Repeating the same steps as in the case with $E > V_0$, we find:

$R = 1 \quad , \quad T = 0$

We therefore have total reflection for $E < V_0$ (similar to the total internal reflection in electromagnetism.

Note, however, that $\psi_{II}(x) \neq 0$ in sharp contrast to the classical situation. This has a very profound consequence. To illustrate, consider the following potential:



For $E < V_0$, we have:

$$\begin{cases} \psi_I(x) = \underbrace{e^{ikx}}_{\psi_I} + \underbrace{A e^{-ikx}}_{\psi_R} & k = \sqrt{\dfrac{2mE}{\hbar^2}} \\ \psi_{II}(x) = B e^{-kx} + C e^{+kx} & k = \sqrt{\dfrac{2m(V_0 - E)}{\hbar^2}} \\ \psi_{III} = \underbrace{D e^{ikx}}_{\psi_T} & \end{cases}$$

Now e^{+kx} term is allowed since it is finite at $x = a$. As a result, $\psi \neq 0$ at $x = a$, which gives rise to an oscillatory solution for $x > a$. Hence, the particle can climb over the barrier and travel to $+\infty$, even though $E < V_0$! This is not possible in classical mechanics.

This phenomenon is called "tunneling". It has very important applications, which we will return to later.