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Square Well Potential (Cont'd):

An important aspect is the discrete nature of energy eigenvalues for $E < V_0$. This always happens for bound states, i.e. states where $\psi_{(x)} \to 0$ as $x \to \pm\infty$. The spectrum of bound states is therefore "quantized". This we also saw in the case of particle in the box.

Now lets consider states with $E > V_0$. In this case, the most general solution (up to an overall normalization constant) is:

$$\begin{cases} \psi_{I}(x) = e^{ik'x} + A e^{-ik'x} \\ \psi_{II}(x) = B e^{ikx} + C e^{-ikx} \\ \psi_{III}(x) = D e^{ik'x} + E e^{-ik'x} \end{cases}$$

$$k' = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

We have four continuity conditions:

$$\left\{\begin{array}{l} \psi_{I}(x=-a) = \psi_{II}(x=-a) \\ \dfrac{d\psi_{I}}{dx}(x=-a) = \dfrac{d\psi_{II}}{dx}(x=-a) \\ \psi_{II}(x=+a) = \psi_{III}(x=+a) \\ \dfrac{d\psi_{II}}{dx}(x=+a) = \dfrac{d\psi_{III}}{dx}(x=+a) \end{array}\right.$$

For any given value of $E > V_0$, we find four linear algebraic equation to determine five constants A, B, C, D, E. This system of equations always has solutions.

Therefore the spectrum for $E > V_0$ is continuous.

Note that states with $E > V_0$ are not bound states since $\psi(x)$ does not tend to 0 as $x \to \pm\infty$ (due to oscillatory behavior of ψ in regions I, III).

It is true in general that the spectrum of unbound states is continuous.

Another statement is that one-dimensional bound states have no degeneracy. To show this, assume that there are two solutions ψ_1, ψ_2 with the same energy E ($\psi_1, \psi_2 \to 0$ as $x \to \pm\infty$):

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + V\psi_1 = E\psi_1$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + V\psi_2 = E\psi_2$$

Multiplying the first equation by ψ_2 and the second one by ψ_1, and then subtracting, we get:

$$\psi_1 \frac{d^2\psi_2}{dx^2} - \psi_2 \frac{d^2\psi_1}{dx^2} = 0 \Rightarrow \frac{d}{dx}\left(\psi_1 \frac{d\psi_2}{dx} - \psi_2 \frac{d\psi_1}{dx}\right) =$$

Thus,

$$\psi_1 \frac{d\psi_2}{dx^2} - \psi_2 \frac{d\psi_1}{dx} = \text{Const.}$$

The constant must be 0 since $\psi_1, \psi_2 \to 0$ as $x \to \pm\infty$. Then:

$$\psi_1 \frac{d\psi_2}{dx} = \psi_2 \frac{d\psi_1}{dx} \Rightarrow \frac{d\psi_1}{\psi_1} = \frac{d\psi_2}{\psi_2} \Rightarrow \psi_1 = e^{\alpha} \psi_2$$

For normalized states $|e^{\alpha}| = 1$, and hence the difference between ψ_1 and ψ_2 is a pure phase. Therefore ψ_1 and ψ_2 are not really different states, they rather represent the same state.

This can be also understood physically. There is degeneracy when there exists a continuous symmetry. In one dimension, the only continuous symmetry is translation. Translation invariance requires that $V_{(x)}$ be a constant. But there will be no bound states for $V = \text{Const}$, in fact we require $V_{(x)}$ not be a constant in order to have bound states.

On the other hand in two or three dimensions there are continuous symmetries for $V \neq$ const. In two dimensions, consider the analogue of a square well potential:

$$\begin{cases} V_{(\rho)} = V_0 & \rho < a \\ V_{(\rho)} = 0 & \rho \geq a \end{cases}$$

We have rotation invariance $\theta \to \theta + \alpha$ for this potential. Similarly, in three dimensions for

$$\begin{cases} V_{(r)} = V_0 & r < a \\ V_{(r)} = 0 & r \geq a \end{cases}$$

we have rotation invariance in three dimensions. This is the reason why bound states in more than one dimension can have degeneracy (e.g., Hydrogen atom, harmonic oscillator).