PHYC 521: Graduate Quantum Mechanics I

Fall 2009

Homework Assignment #4

(Due October 7)

1- Consider a real localized potential in one dimension $V(x)$ that vanishes outside some region $[a, b]$. Outside this region the scattering eigenstates are forward and backward propagating plane waves:

$$\begin{aligned} \psi(x) &= Ae^{ikx} + Be^{-ikx} \qquad x < a \\ \psi(x) &= Ce^{ikx} + De^{-ikx} \qquad x > b\,. \end{aligned}$$

The scattering matrix, known for short as the "S-matrix", relates the incoming and outgoing waves at a fixed energy $E = \hbar^2 k^2/2m$:

$$\begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} A \\ D \end{bmatrix}$$

(a) Show that conservation of probability implies that the S-matrix is unitary, and thus its eigenvalues are of the form $e^{i\phi_1}, e^{i\phi_2}$ with real ϕ_1, ϕ_2 known as the "scattering phase shifts".

(b) For the particular case that $V(x)$ has reflection symmetry, the S-matrix takes the form

$$S = \begin{bmatrix} r & t \\ t & r \end{bmatrix}$$

where r and t are the complex reflection and transmission amplitudes, with the reflection and transmission probabilities $R = |r|^2$ and $T = |t|^2$. Using the results from part (a) show that r, t are given in terms of two real parameters R and ϕ_r

$$r = \sqrt{R}e^{i\phi_r} \quad , \quad t = i\sqrt{1-R}e^{i\phi_r}\,.$$

Relate r, t and R, ϕ_r to scattering phase shifts ϕ_1, ϕ_2.

2- A coherent state represents the closest quantum-mechanical wavepacket to a classical motion. It is constructed from the energy eigenstates of a harmonic oscillator as follows:

$$|\psi\rangle = \exp(-|c|^2/2) \sum_{n=0}^{\infty} \frac{c^n}{\sqrt{n!}} |n\rangle \, ,$$

where c is an arbitrary complex number.

(a) Show that $|\psi\rangle$ is an eigenstate of the lowering operator $\mathbf{a}$ and find the corresponding eigenvalue.

(b) Find the expectation value and uncertainty in the number operator $N = \mathbf{a}^\dagger \mathbf{a}$ and show that $\Delta N / \langle N \rangle \to 0$ as $\langle N \rangle \to \infty$.

3- Exercise 7.4.3, Shankar, 2nd edition, page 212.

4- Exercise 7.4.6, Shankar, 2nd edition, page 212.