Instructions:
• The exam consists of 10 short-answer problems (10 points each).
• Where possible, show all work; partial credit will be given if merited.
• Personal notes on two sides of an 8×11 page are allowed.
• Total time: 3 hours.

Unless otherwise noted, commonly used symbols are defined as follows:

\[ P \quad \text{Pressure} \]
\[ T \quad \text{Absolute Temperature} \]
\[ k \quad \text{Boltzmann’s constant} \]
\[ R \quad \text{Gas constant} \]
\[ \hbar \quad \text{Planck’s constant divided by } 2\pi \]
\[ \beta \quad (kT)^{-1} \]
\[ S \quad \text{Entropy} \]
\[ V \quad \text{Volume} \]
\[ \mu \quad \text{Chemical potential} \]

Useful Integrals and Sums:

\[ \int_{-\infty}^{\infty} dx e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}; \ \text{Re} \alpha > 0 \]
\[ \int dx e^{-x} = -e^{-x} + C \]
\[ \sum_{m=0}^{\infty} x^m = \frac{1}{1-x}; \ |x| < 1. \]
P1. An isolated system consisting of $N$ noninteracting, distinguishable, spin-$1/2$ particles is in equilibrium in a uniform magnetic field of strength $B$. Of these, exactly $N_1$ spins are aligned with the field and $N - N_1$ spins are aligned against the field, so that the total energy $E = -N_1\mu B + (N - N_1)\mu B$, where $\mu$ is the magnetic moment of each particle. Obtain an expression for the temperature of the system as a function of $N$, $N_1$, and $B$. Are there any restrictions on $N_1$?

P2. A dilute gas having $N$ atoms of mass $m$ is confined to a cylinder with radius $r$ and length $L$ and in equilibrium at a temperature $T$. The length is controlled by a piston which is held in place by a couple of pins. The bottom face of the cylinder and the head of the piston comprise the two electrodes of a parallel plate capacitor. These surfaces are given the charges $Q$ and $-Q$ respectively, and their mutual attraction that compresses the gas in the cylinder. (See figure.) The free energy for constant $Q$, $L$, $T$ and $N$ is given by

$$F = -NkT \ln \left( \frac{\pi r^2 L}{N} \right) - \frac{3}{2}NkT \ln \left( \frac{mkT}{2\pi \hbar^2} \right) + \frac{1}{2} \left( \frac{L}{\epsilon_0 \pi r^2} \right) Q^2,$$

where $\epsilon_0$ is the permittivity of free space. If the pins are removed and the piston is allowed to slide freely, what will be the value of $L$ when the system reaches equilibrium?
**P3.** A gas of $N$ noninteracting ions (ideal gas) is in thermal equilibrium at a temperature $T$ in a vessel of volume $V$. The vessel is divided in half by a rigid membrane, but there are small channels in the membrane through which the ions can pass. (See figure.) Within a channel, an ion experiences an accelerating electric field such that the energy to be on one side of membrane is higher than the energy to be on the other side by an amount $\Delta$. Show that the difference in pressure $\delta P$ across the membrane is given by

$$\delta P = \frac{2NkT}{V} \tanh \left( \frac{\Delta}{2kT} \right).$$

**P4.** Many of the electronic properties of a metal may be understood through a model in which the mobile electrons in the conduction band are treated as a gas of noninteracting indistinguishable particles having spin $1/2$ and mass $m$. For a typical metal, the conduction band electron density is $n = 30 \times 10^{27}$ m$^{-3}$, and $m = 0.51$ MeV/c$^2$ (9.1 $\times 10^{-31}$ Kg). For these material parameters, find the speed (in m/s) of the fastest conduciton electrons in the low temperature ($T \to 0$) limit.
P5. Consider a one-dimensional gas of \( N \) point-particles, each having mass \( m \), confined to a line of length \( L \). (See figure.) The particles move freely except when in contact with each other, and then their potential energy is infinite; this prevents them from changing places on the line. The leftmost and rightmost particles recoil elastically from the end stops, which are at fixed positions. By integrating over (classical) phase space, show that the canonical partition function is

\[
Z = \left( \frac{\sqrt{mkT}}{2\pi \hbar^2} \right)^N \frac{L^N}{N!}.
\]

Find the average force \( f \) on the end stops, as a function of \( L \) and \( T \).

\[\text{[Image]}\]

P6. The fundamental equation for the thermodynamic properties of a rubber band is given by

\[
S = S_0 + cL_0 \ln \frac{U}{U_0} - \frac{b(L - L_0)^2}{2(L_1 - L_0)},
\]

where \( S \) is the entropy, \( U \) is the internal energy, and \( L \) is the length of the rubber band, and \( c, b, U_0, \) and \( S_0 \) are positive constants. The unstretched length is \( L_0 \), and the length at the elastic limit is \( L_1 \). Suppose that a rubber band in equilibrium at a temperature \( T \) is allowed to contract isothermally and reversibly, from \( L = L_1 \) to \( L = L_0 \). How much heat will be absorbed from the surroundings during this process?

P7. A system at constant volume is in equilibrium with a particle reservoir at a temperature \( T \). The energy of the system is proportional to the number of particles; when the system contains \( n \) particles, the system’s energy \( E = n\Delta \), where the incremental energy \( \Delta = 2.5 \text{ meV} \). The chemical potential of the reservoir is \( 1.25 \text{ meV} \). Evaluate the grand canonical partition function

\[
Z = \sum_i e^{-\beta E_i} e^{\beta \mu N_i}
\]

and show that \( \langle n \rangle = 200 \) when \( T = 300 \text{ K} \).
P8. A system contains $N$ noninteracting particles, each of which can be in one of two states having energies $\varepsilon = 0$ and $\varepsilon = \Delta$ respectively. The system is in equilibrium with the surroundings at a temperature $T$. Calculate the energy average, $\langle E \rangle$, and the rms deviation, $\delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2}$. How does the ratio $\delta E / \langle E \rangle$ depend on the size of the system, and what is its significance to thermodynamics?

P9. If the temperature of the atmosphere is 5°C on a winter day and if 1 kg of water at 90°C is available, what is the maximum amount of work that can be extracted (assuming that one is clever enough to figure out a way to do so) as the water is cooled to the ambient temperature? Assume that the volume of water is constant, and assume that the molar heat capacity at constant volume is 4.17 kJ kg$^{-1}$K$^{-1}$ and is independent of temperature.

P10. For audible frequencies, the speed of sound in air is given by
\[ v = \sqrt{\frac{B_S}{\rho}} \]
where $\rho$ is the mass density and
\[ B_S = -V \left( \frac{\partial P}{\partial V} \right)_S \]
is the adiabatic bulk modulus. Assuming ideal gas behavior ($PV = nRT$), with a molar heat capacity at constant volume $C_V = \frac{5}{2}RT$, calculate the adiabatic bulk modulus and show that
\[ v = \sqrt{\frac{7RT}{5A}}, \]
where $A$ is the average molecular weight (grams/mole) of an air molecule.