Instructions:
- The exam consists of 10 problems (10 points each).
- Where possible, show all work; partial credit will be given if merited.
- Useful formulae are provided below; crib sheets are not allowed.

Useful Formulae:

\[
E = \left( n + \frac{1}{2} \right) \hbar \omega \; ; \; \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (b + b^\dagger) \; ; \; \hat{p} = -i \sqrt{\frac{\hbar m\omega}{2}} (b - b^\dagger)
\]

\[
\psi_n(x) = \frac{1}{\sqrt{2^n n! (\pi x_0^2)}} e^{-x_0^2/2} H_n \left( x / x_0 \right) ; \; x_0 = \sqrt{\frac{\hbar}{m\omega}} \; ; \; H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})
\]

\[
\left[ \hat{S}_x, \hat{S}_y \right] = i\hbar \hat{S}_z \; , \; D(\hat{n}, \theta) = \exp \left( -i \frac{\theta}{\hbar} \hat{n} \cdot \hat{S} \right)
\]

\[
\psi_{100} = \frac{1}{\sqrt{4\pi}} \times \frac{2}{\alpha_0 \sqrt{\alpha_0}} e^{-r/\alpha_0}
\]

\[
\psi_{200} = \frac{1}{\sqrt{4\pi}} \times \frac{1}{2 \sqrt{2} \alpha_0 \sqrt{\alpha_0}} (2 - \frac{r}{\alpha_0}) e^{-r/2\alpha_0}
\]

\[
\psi_{2\alpha} = \frac{3}{\sqrt{4\pi}} \cos \theta \times \frac{1}{2 \sqrt{6} \alpha_0 \sqrt{\alpha_0}} (\frac{r}{\alpha_0}) e^{-r/2\alpha_0}
\]

\[
\psi_{2\alpha} = \frac{\pm}{\sqrt{8\pi}} \sin \theta e^{\pm r/\alpha_0} \times \frac{1}{2 \sqrt{6} \alpha_0 \sqrt{\alpha_0}} (\frac{r}{\alpha_0}) e^{-r/2\alpha_0}
\]

\[
\psi_{3\alpha} = \frac{1}{\sqrt{4\pi}} \times \frac{2}{2\sqrt{3} \alpha_0 \sqrt{\alpha_0}} (27 - 18 \frac{r}{\alpha_0} + 2 \frac{r^2}{\alpha_0^2}) e^{-r/3\alpha_0}
\]

\[
1 \text{eV} = 1.6 \times 10^{-19} \text{ J}
\]

\[
\frac{e^2}{hc} = \frac{1}{137} \; ; \; \hbar c = 197 \text{MeV} \cdot \text{fm}
\]

\[
\int_{-\infty}^{\infty} dx x^2 \exp(-ax^2) = \frac{1}{\sqrt{4 \pi a^3}}
\]

\[
\int_{-\infty}^{\infty} dx \exp(-ax^2) \exp(ikx) = \sqrt{\pi / a} \exp(\frac{-a k^2}{4})
\]

\[
\int_{-\infty}^{\infty} dx \left( x^2 + a^2 \right)^{-1} \exp(ikx) = \frac{\pi}{a} \exp(-a |k|)
\]
P1. A system consists of two spins, $\vec{s}_1$ and $\vec{s}_2$ in a uniform magnetic field $\vec{B} = B_0 \hat{z}$ directed along the z axis. Their evolution is governed by the Hamiltonian,

$$H = \frac{\varepsilon}{\hbar} (\vec{s}_1 \cdot \vec{s}_2) - \frac{\mu}{\hbar} \vec{B} \cdot (\vec{s}_1 + \vec{s}_2)$$

Which of the following quantities are constants of the motion: $\vec{s}_1, \vec{s}_2, \vec{s}_1^2, \vec{s}_2^2, \vec{s} = \vec{s}_1 + \vec{s}_2, \vec{s}_z$? If both are spin-1/2, what are the eigenvectors and eigenvalues of $H$?

P2. Consider a spin-1 particle. The matrix representation of the components of spin-1 angular momentum are,

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \hat{S}_y = \frac{\hbar}{i \sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \hat{S}_z = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The matrices are expressed in the ordered basis $\{|+1\rangle, |0\rangle, |-1\rangle\}$ where the kets are labeled by the eigenvalues of $\hat{S}_z$ in units of $\hbar$.

Show that $|\psi_A\rangle = \frac{1}{2} \left(|+1\rangle + \sqrt{2} |0\rangle + |-1\rangle\right)$, $|\psi_B\rangle = \frac{1}{2} \left(|+1\rangle - \sqrt{2} |0\rangle + |-1\rangle\right)$, and $|\psi_C\rangle = \frac{1}{\sqrt{2}} \left(|+1\rangle - |-1\rangle\right)$ are eigenvectors of $\hat{S}_z$, and find their eigenvalues. When a beam of spin-1 particles initially prepared in the state $|+1\rangle$ is put through a Stern-Gerlach apparatus having the magnetic field gradient in the x-direction, three beams emerge. What are the relative intensities of the three beams?
P3. A deuteron is a bound state of a proton and a neutron, \( m_N = m_p \approx 939 \text{ MeV/c}^2 \). The attraction between the two particles can be modeled by a square-well potential of width \( b \) and depth \( -V_0 \), as illustrated in the figure.

\[
\begin{align*}
V(r) & \quad \text{(V(r))} \\
& \\
\end{align*}
\]

The reduced radial wavefunction \( u(r) = rR(r) \) for the ground state is determined from the time-independent Schrödinger equation for motion relative to the center of mass,

\[
-\frac{\hbar^2}{2\mu} \frac{d^2 u}{dr^2} + V(r)u = Eu
\]

Assume a solution of the form \( u(r) = A\sin(kr) \) for the region \( 0 < r < b \), and \( u(r) = B\exp(-qr) \) for \( b \leq r \), where \( A, B \) are normalization constants; find the relations for \( k \) and \( q \) in terms of \( E \), and show that \( E \) is determined by the transcendental equation \( \tan(kb) = -k/q \). If \( V_0 = 38.5 \text{ MeV} \), what is the minimum width \( b \) (in fm) below which there are no bound states?

(Note: \( \hbar c = 197 \text{ MeV-fm} \))

P4. The wavefunction for a particle moving freely on the x-axis is given by

\[
\psi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{x^2}{4\sigma^2}\right) \exp(ik_0x)
\]

where \( \sigma \) and \( k_0 \) are constants. Calculate the distribution of expected outcomes of a measurement of the particle’s energy \( E \). What is the mean of this distribution?

P5. Consider a particle moving in three dimensions. Show which of the following observables can be measured simultaneously:

\[
L_x = yp_z - zp_y, L_y = zp_x - xp_z, L_z = xp_y - yp_x, z, p_x, r^2 = x^2 + y^2 + z^2
\]

Here \((x, y, z)\) and \((p_x, p_y, p_z)\) are the three Cartesian components of position and momentum respectively, and \( r \) is the radius from the origin.
P6. A simple three-parameter potential that is often used to characterize the atomic separation distance $r$ in a diatomic molecule is the Morse potential, given by

$$U(r) = D \left(1 - \exp\left(-a(r-r_0)\right)\right)^2,$$

and sketched in the graph. Here $r_0$ is the equilibrium bond distance, $D$ is the dissociation energy, and $a$ is a parameter that determines the width of the potential. For the $N_2$ molecule in the ground state, the equilibrium bond distance is 1.0977 angstroms, and the dissociation energy is 9.79 eV. For a zero-point vibrational energy of 0.146 eV, what is the value of $a$ (in angstroms)? The atomic weight of N is 14.0067 grams per mole ($2.33 \times 10^{-26}$ kg per atom).

P7. The hyperfine splitting of the ground state in hydrogen is due to the interaction of electron spin $\vec{s}$ and the proton spin $\vec{i}$, and can be described by the interaction Hamiltonian,

$$H_{\text{int}} = A \delta(\vec{r}) (\vec{s} \cdot \vec{i}),$$

where the constant $A$ depends on the electron and proton magnetic moments and $\delta(\vec{r}) = \delta(x) \delta(y) \delta(z)$ is a delta-function in three dimensions. Using the hydrogen wave functions provided in the table, obtain an expression for the hyperfine splitting in terms of $A$ and the Bohr radius $a_0$, to lowest order in $H_{\text{int}}$. 
P8. Consider a two-port interferometer with 50/50 beam splitters, each arm having a path length \( L \). The optical path for one arm can be made longer through a phase shift \( \theta \). A single photon of frequency \( \omega \) is prepared and injected into the input port. If one of the two mirrors is only partially reflecting, with a reflection probability (reflectance) \( R \leq 1 \) as indicated in the figure below, what is the probability of detecting the photon at the denoted output-port versus \( \theta \) and \( R \)?

![Interferometer Diagram]

P9. A particle with mass \( m \) is trapped in the ground state of a harmonic oscillator well described by the potential energy function \( U_1(x) = \frac{1}{2} kx^2 \). Suddenly the particle is subjected to a constant force \( F \), so that the new potential well is given by \( U_2(x) = \frac{1}{2} kx^2 - Fx \). What is the probability that the particle will be in the ground state of the new well?

P10. The eigenstates \( |n\rangle \) of the harmonic oscillator Hamiltonian \( H = \hbar \omega (b^\dagger b + 1/2) \) are labeled by the quantum number \( n \), with energy eigenvalue \( E_n = \hbar \omega (n + 1/2) \). A coherent state,

\[
|\lambda\rangle = e^{-|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle
\]

is a superposition of these eigenstates, weighted respectively by \( \frac{\lambda^n}{\sqrt{n!}} \), where \( \lambda \) is any complex number. Coherent states have the property that they are eigenstates of the lowering-operator, i.e., \( b|\lambda\rangle = \lambda |\lambda\rangle \). Consider the time evolution of a coherent state. If the initial state \( |\psi(0)\rangle = |\lambda\rangle \), where \( \lambda = e^{i\phi} \) show that \( |\psi(t)\rangle = e^{-i\epsilon t/\hbar} |\lambda\rangle \) where \( \epsilon = \sqrt{2m \omega} \). Use this result to show that the expectation value of the position evolves according to \( \langle x(t) \rangle = x_0 \cos(\omega t - \phi) \), where \( x_0 = \frac{\sqrt{2m \omega}}{\hbar} \).