10.21 ** Let us start from the given expression and show that its elements are equal to the usual definitions (10.37) and (10.38). First

\[ I_{xx} = \int \varrho (r^2 - xx) dV = \int \varrho (x^2 + y^2) dV \]

which is exactly the definition (10.37) (in integral form). The two other diagonal elements work the same way. Meanwhile,

\[ I_{xy} = \int \varrho (0 - xy) dV = -\int \varrho (xy) dV \]

which is exactly the definition (10.38). The other off-diagonal elements all work the same way, and we've shown that the proposed definition agrees with the original one.

10.24 ** (a) For rotation about \( P \), the moment of inertia

\[ I_{xx} = \sum m_\alpha (y_\alpha^2 + z_\alpha^2) \]

From the picture, you can see that \( r_\alpha = r'_\alpha - \Delta \), so that \( x_\alpha = x'_\alpha - \xi \) and so on.

Therefore

\[ I_{xx} = \sum m_\alpha [(y'_\alpha - \eta)^2 + (z'_\alpha - \zeta)^2] = \sum m_\alpha (y'_\alpha^2 + z'_\alpha^2) + \sum m_\alpha (\eta^2 + \zeta^2) - 2\eta \sum m_\alpha y'_\alpha - 2\zeta \sum m_\alpha z'_\alpha. \]

The first sum on the second line is just \( I_{xx}^{\text{cm}} \). The second is \( M(\eta^2 + \zeta^2) \), and the last two are zero by (10.7). Thus

\[ I_{xx} = I_{xx}^{\text{cm}} + M(\eta^2 + \zeta^2) \quad (\text{iv}) \]

as claimed. The other two diagonal elements work the same way, as do the six off-diagonal terms; for instance,

\[ I_{yz} = I_{yz}^{\text{cm}} - M\eta\zeta. \quad (\text{v}) \]

(b) In Example 10.2(b) we found \( \mathbf{I}^{\text{cm}} \) for a cube in (10.52), which gives

\[ I_{xx}^{\text{cm}} = \frac{1}{6} M a^2 \quad \text{and} \quad I_{yz}^{\text{cm}} = 0. \]

In part (a) of the same example, we found \( \mathbf{I} \) for the same cube rotating about a corner, which is displaced from the CM by \( \Delta = (-a/2, -a/2, -a/2) \). There we found in (10.49)

\[ I_{xx} = \frac{2}{3} M a^2 = \frac{1}{6} M a^2 + 2M(-a/2)^2 \quad \text{and} \quad I_{yz} = -\frac{1}{4} M a^2 = 0 - M(-a/2)(-a/2). \]

As you can easily see these are precisely the relations (iv) and (v) with \( \eta = \zeta = -a/2 \).
10.29 * That \( O \mathbf{x} \) is a principal axis means that if \( \omega \) is along \( O \mathbf{x} \) then \( \mathbf{L} \) is also along \( O \mathbf{x} \). If we think in terms of matrices, this says that if \( \omega \) is a column with entries \( \omega_1, 0, 0 \), then \( \mathbf{L} = I \omega \) is a column whose second and third entries are also zero. This requires that \( I_{yz} = I_{xz} = 0 \). Similarly, that \( O \mathbf{y} \) and \( O \mathbf{z} \) are principal axes requires that \( I_{xy} = I_{yx} = 0 \) and \( I_{xz} = I_{yz} = 0 \). This leaves \( I \) with \( I_{xx}, I_{yy}, \) and \( I_{zz} \) down the diagonal and zeroes everywhere else.

10.36 ** (a) The three masses are equal, \( m_1 = m_2 = m_3 = m \) and their positions are 
\[
\mathbf{r}_1 = a(1, 0, 0), \quad \mathbf{r}_2 = a(0, 1, 2), \quad \text{and} \quad \mathbf{r}_3 = a(0, 2, 1).
\]
Therefore
\[
\begin{align*}
I_{xx} &= \sum m_\alpha (y_\alpha^2 + z_\alpha^2) = ma^2(0 + 5 + 5) = 10ma^2 \\
I_{yy} &= \sum m_\alpha (x_\alpha^2 + z_\alpha^2) = ma^2(1 + 4 + 1) = 6ma^2 \\
I_{zz} &= \sum m_\alpha (x_\alpha^2 + y_\alpha^2) = ma^2(1 + 1 + 4) = 6ma^2 \\
I_{xy} &= -\sum m_\alpha x_\alpha y_\alpha = -ma^2(0 + 0 + 0) = 0 \\
I_{xz} &= -\sum m_\alpha x_\alpha z_\alpha = -ma^2(0 + 0 + 0) = 0 \\
I_{yz} &= -\sum m_\alpha y_\alpha z_\alpha = -ma^2(0 + 2 + 2) = -4ma^2
\end{align*}
\]
\[
\text{or } \mathbf{I} = 2ma^2 \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}
\]

(b) As you can check, the characteristic equation is
\[
\det(\mathbf{I} - \lambda \mathbf{1}) = (10ma^2 - \lambda)^2(2ma^2 - \lambda) = 0
\]
Therefore, the principal moments are \( \lambda_1 = \lambda_2 = 10ma^2 \) and \( \lambda_3 = 2ma^2 \). If we set \( \lambda = 10ma^2 \), the equation \( (\mathbf{I} - \lambda \mathbf{1})\omega = 0 \) yields three equations, \( \omega_1 = 0, \omega_2 + \omega_3 = 0, \) and \( \omega_2 + \omega_3 = 0 \), of which only one is independent. Thus there are two independent eigenvectors with \( \lambda = 10ma^2 \), which we can take to be \( \mathbf{e}_1 = (1, 0, 0) \) and \( \mathbf{e}_2 = (0, 1, -1)/\sqrt{2} \) or any other two perpendicular directions in the plane of these two. If we set \( \lambda = 2ma^2 \), the equation \( (\mathbf{I} - \lambda \mathbf{1})\omega = 0 \) yields three equations, \( \omega_1 = 0, \omega_2 - \omega_3 = 0, \) and \( -\omega_2 + \omega_3 = 0 \). There is just one independent eigenvector with \( \lambda = 2ma^2 \), which we can take to be \( \mathbf{e}_3 = (0, 1, 1)/\sqrt{2} \).

10.40 ** (a) Multiplying the first of Equations (10.86) by \( \lambda_1 \omega_1 \), the left side becomes \( \lambda_1^2 \omega_1 \omega_1 \), which is the same as \( \frac{1}{2} d(\lambda_1^2 \omega_1^2)/dt \). Therefore
\[
\frac{d}{dt} (\lambda_1^2 \omega_1^2) = 2\lambda_1 (\lambda_2 - \lambda_3) \omega_1 \omega_2 \omega_3.
\]
Similarly, the second and third equations give
\[
\frac{d}{dt} (\lambda_2^2 \omega_2^2) = 2\lambda_2 (\lambda_3 - \lambda_1) \omega_1 \omega_2 \omega_3 \quad \text{and} \quad \frac{d}{dt} (\lambda_3^2 \omega_3^2) = 2\lambda_3 (\lambda_1 - \lambda_2) \omega_1 \omega_2 \omega_3.
\]
Adding these three equations and remembering that \( \mathbf{L} = (\lambda_1 \omega_1, \lambda_2 \omega_2, \lambda_3 \omega_3) \), we find that
\[
\frac{d}{dt} \mathbf{L}^2/dt = 0.
\]
(b) If, instead, we multiply the first of Equations (10.88) by \( \omega_1 \), we find that
\[
\frac{1}{2} \frac{d}{dt} (\lambda_1 \omega_1^2) = (\lambda_2 - \lambda_3) \omega_1 \omega_2 \omega_3.
\]
Adding this to the corresponding two equations for the second and third components, we find that
\[
\frac{1}{2} \frac{d}{dt} (\lambda_1 \omega_1^2 + \lambda_2 \omega_2^2 + \lambda_3 \omega_3^2) = \frac{d}{dt} T_{rot} = 0.
\]
The inertia tensor for the book (sides \( a = 30 \), \( b = 20 \), and \( c = 3 \), all in cm) can be evaluated as in Example 10.2. With the origin at the CM, all off-diagonal elements are zero, and the diagonal elements (which are the principal moments) are \( \lambda_1 = M(b^2 + c^2)/12 \), and so on. If the book's spin axis is close to the shortest symmetry axis (the \( z \) axis), then according to (10.91) the frequency of wobble is given by

\[
\Omega^2 = \frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_2} \omega_3^2 = \frac{(a^2 + b^2 - b^2 - c^2)(a^2 + b^2 - c^2 - a^2)}{(b^2 + c^2)(c^2 + a^2)} \omega_3^2
\]

\[
= \frac{(a^2 - c^2)(b^2 - c^2)}{(a^2 + c^2)(b^2 + c^2)} \omega_3^2 \tag{xvii}
\]

Putting in the given numbers, we find \( \Omega = 0.968 \omega_3 = 174 \text{ rpm} \). If the book is spinning about the longest (\( x \)) axis we have only to swap \( \lambda_1 \) and \( \lambda_3 \), and we find \( \Omega = 0.614 \omega_3 = 111 \text{ rpm} \).