\[ \frac{-\hbar^2}{2\mu} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) - \frac{\hbar^2}{2\mu r^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] + V(r)\psi = E\psi \] 

7.9

\[ (L_{op})^2 \psi \]

\[ \frac{\left( L_{op} \right)^2 \psi}{2\mu r^2} \]

with:

\[ (L_{op})^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] \]

7.20

\[ \left( P_{op} \right)^2 = -\hbar^2 \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) \]

7.19

then "if" \( \left( P_{op} \right)^2 \rightarrow P_r^2 \psi \)

with \( P_r = \text{radial linear momentum} \)

\( \left( L_{op} \right)^2 \psi \rightarrow L^2 \psi \)

with \( L = \text{angular momentum} \)

\( (0 \leq \phi \leq 2\pi) \)

then \( (7.9) \) is equivalent to:

\[ \left( \frac{P_r^2}{2m} + \frac{L^2}{2\mu r^2} + V(r) = E \right) \psi \]
with \( \frac{p_r^2}{2m} \) = KE of radial motion

\[ \frac{1}{2m} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{2\mu r^2}{h^2} \left[ E - V(r) \right] = -\left[ \frac{1}{f(\theta)\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\theta}{d\theta} \right) + \frac{1}{g(\phi)\sin^2 \theta} \frac{d^2 g(\phi)}{d\phi^2} \right] \]

RHS:

\[ -\left[ \frac{g(\phi)}{f(\theta)g(\phi)\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \left( \sin \theta \frac{\partial}{\partial \phi} \right) \right] = \ell (\ell + 1) \frac{f(\theta)g(\phi)}{f(\theta)g(\phi)} \]

and \( f(\theta)g(\phi) = Y_{\ell m}(\theta, \phi) \) thus:

\[ -\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right] Y_{\ell m}(\theta, \phi) = \ell (\ell + 1) Y_{\ell m}(\theta, \phi) \]

\[ -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right] = \ell (\ell + 1) \hbar^2 Y_{\ell m}(\theta, \phi) \]

\[ \ell (\ell + 1) \hbar^2 Y_{\ell m}(\theta, \phi) = \ell (\ell + 1) \hbar^2 Y_{\ell m}(\theta, \phi) \]

\( \ell \) and \( \ell (\ell + 1) \) are quantum numbers.

\( \ell \) is a non-negative integer.

\( \ell (\ell + 1) \) is proportional to the angular momentum quantum number \( l \).

\( \ell (\ell + 1) \) is the angular momentum quantum number squared.

\( \ell (\ell + 1) \) is the total angular momentum quantum number squared.

\[ \ell (\ell + 1) Y_{\ell m}(\theta, \phi) = \ell (\ell + 1) \hbar^2 Y_{\ell m}(\theta, \phi) \]

\[ Y_{\ell m}(\theta, \phi) \text{ is a spherical harmonic.} \]
So the $Y_{\ell m}$ solutions to the RHS of eqn 7-12 are so-called eigenfunctions of the differential operator $(\nabla^2 \phi)$ with eigenvalues $\ell (\ell + 1)$.

The $Y_{\ell m}$'s are also eigenfunctions of the differential operator:

$$L_{z_{op}} Y_{\ell m} = m \hbar Y_{\ell m}$$

And $Y_{n\ell m} = R_{n\ell} Y_{\ell m}$ are eigenfunctions of the "Hamiltonian" operator eqn 7-9:

$$-\frac{\hbar^2}{2m} \nabla^2 \phi + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \phi \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \phi + V(\rho) Y_{n\ell m} = E_n Y_{n\ell m}$$
Notes:

1) While there is a \((\text{Prop})^2\) such that
\[ (\text{Prop})^2 \rightarrow p_r^2 \]

\(p_r^2\) is not a constant (value)

thus \(\gamma\), and more specifically \(R_\infty(r)\) is not an eigenfunction of \((\text{Prop})^2\).

2) That \( (\text{Prop})^2 \gamma = \epsilon (\text{L} + \text{h}) \gamma \)

\(\gamma\) is a constant (angular momentum)

and this is a constant

means that \( \langle \text{L}^2 \rangle = \epsilon (\text{L} + \text{h}) \gamma \) (a constant)

which is the same result you will get for central force problems in classical mechanics.

3) Curiously, \( (\text{Prop})^2 \rangle = m \gamma \)

means that \( \langle \text{L}^2 \rangle = m \gamma \) (a constant)

is also a measurable of the system.
4) Most importantly, eigenvalues are measurable because

\[ \langle \text{operator} \rangle = \int \psi^* \text{operator} \psi \, d\text{vol} \]

\[ \left( \text{eg} \, \ell^2, \text{or } L^2 \right) \]

\[ = \text{eigenvalue} \cdot \int \psi^* \psi \, d\text{vol} \]

\[ \text{must be a constant to come out of integral} \]

\[ = \text{eigenvalue} \]

all measurables are obtained by this procedure.