Physics 405
Electricity and Magnetism

Midterm Exam II
Friday, 04/01/2016

Instructions:

• The exam consists of 4 problems. The maximum grade is 25 points.
• You may only use personal notes that fit on ONE 8.5’’ × 11’’ sheet of paper.
• Total time is 100 minutes.

Useful formulas and relations:

• Relation of spherical coordinates, \((r, \theta, \phi)\), to Cartesian coordinates:
  \[ x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta. \]
  Unit basis vectors:
  \[ \hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}, \quad \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}, \quad \hat{\theta} = \hat{\phi} \times \hat{r}. \]
  Volume element: \(dr = r^2 \sin \theta \, d\theta \, d\phi\). Area element: \(da = r \sin \theta \, d\theta \, d\phi\)
  Gradient:
  \[ \vec{\nabla} f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}. \]
  Divergence:
  \[ \vec{\nabla} \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}. \]
  Curl:
  \[ \vec{\nabla} \times \vec{A} = \frac{1}{r \sin \theta} \left( \frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right) \hat{r} + \frac{1}{r} \left( \frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right) \hat{\theta} + \frac{1}{r} \left( \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \hat{z}. \]

• Relation of cylindrical coordinates, \((s, \phi, z)\), to Cartesian coordinates:
  \[ x = s \cos \phi, \quad y = s \sin \phi, \quad z = z. \]
  Unit basis vectors:
  \[ \hat{s} = \cos \phi \hat{x} + \sin \phi \hat{y}, \quad \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}. \]
  Volume element: \(ds = ds \, d\phi \, dz\). Area element in the \(xy\) plane: \(da = ds \, d\phi\)
  Gradient:
  \[ \vec{\nabla} f = \frac{\partial f}{\partial s} \hat{s} + \frac{1}{s} \frac{\partial f}{\partial \phi} \hat{\phi} + \frac{\partial f}{\partial z} \hat{z}. \]
  Divergence:
  \[ \vec{\nabla} \cdot \vec{A} = \frac{1}{s} \frac{\partial}{\partial s} (s A_s) + \frac{1}{s} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}. \]
  Curl:
  \[ \vec{\nabla} \times \vec{A} = \left( \frac{1}{s} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{s} + \left( \frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s} \right) \hat{\phi} + \frac{1}{s} \left( \frac{\partial (s A_\phi)}{\partial s} - \frac{\partial A_s}{\partial \phi} \right) \hat{z}. \]
1. A point charge \(-q\) is placed at distance \(d\) from the center of a conducting sphere of radius \(R\), as shown in the figure.

(a) [3 points] Find the force between the sphere and the point charge if the sphere is grounded.

(b) [3 points] Find the force between the sphere and the point charge if the sphere is neutral and isolated (this means that it is not connected to any source of potential and has no charge).

![Diagram](image)

2. [6 points] The potential at the surface of a sphere (radius \(R\)) is given by \(V(\theta) = V_0 \sin^2(\theta/2)\), where \(V_0\) is a constant. Find the potential outside the sphere, that is for \(r \geq R\).

\[ V(\theta) = V_0 \sin^2(\theta/2) \]

\[ V_0 \]

Some expressions you may find useful: \(\sin^2(x) = \frac{1}{2} [1 - \cos(2x)]\), \(P_0(\cos \theta) = 1\), and \(P_1(\cos \theta) = \cos \theta\), where \(P_0\) and \(P_1\) are the first two Legendre polynomials.

3. A sphere of radius \(R\) carries a charge distribution \(\rho = k \cos(\theta)/r\), where \(k\) is a constant.

(a) [3 points] Find the monopole moment of this charge distribution.

(b) [3 points] Find the dipole moment of this charge distribution.

4. The space between the plates of a parallel-plate capacitor is filled with two slabs of dielectric material with permittivities \(\varepsilon_1\) and \(\varepsilon_2\), and thicknesses \(d_1\) and \(d_2\), as shown in the figure. The capacitor is connected to a battery in such way that there is a potential difference \(V\) between the plates, as indicated in the figure (the lower plate is at higher potential than the upper one).

(a) [2 points] Find the magnitude and direction of \(\vec{D}\) within the capacitor plates.

(b) [2 points] Find the magnitude and direction of \(\vec{E}\) within the capacitor plates.

(c) [2 points] Find the magnitude and direction of \(\vec{P}\) within the capacitor plates.

(d) [1 point] Find the bound charge densities for the upper dielectric (slab with \(\varepsilon_1\)).