Physics 405
Electricity and Magnetism

Midterm Exam I
Friday, 02/19/2016

Instructions:

• The exam consists of 4 problems. The maximum grade is 25 points.

• You may only use personal notes that fit on ONE 8.5″ × 11″ sheet of paper.

• Total time is 70 minutes.

Useful formulas and relations:

• Relation of spherical coordinates, \((r, \theta, \phi)\), to Cartesian coordinates:
  \[ x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta. \]

Unit basis vectors:
  \[ \hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}, \quad \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}, \quad \hat{\theta} = \hat{\phi} \times \hat{r}. \]

Volume element: \(d\tau = r^2 dr \sin \theta d\theta d\phi\). Area element: \(da = r^2 \sin \theta d\theta d\phi\)

Gradient:
  \[ \vec{\nabla} f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}. \]

Divergence:
  \[ \vec{\nabla} \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}. \]

Curl:
  \[ \vec{\nabla} \times \vec{A} = \left( \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right) \hat{r} + \frac{1}{r} \left( \frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right) \hat{\theta} + \frac{1}{r} \left( \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \phi} \right) \hat{\phi}. \]

• Relation of cylindrical coordinates, \((s, \phi, z)\), to Cartesian coordinates:
  \[ x = s \cos \phi, \quad y = s \sin \phi, \quad z = z. \]

Unit basis vectors:
  \[ \hat{s} = \cos \phi \hat{x} + \sin \phi \hat{y}, \quad \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}. \]

Volume element: \(d\tau = ds d\phi dz\). Area element in the \(xy\) plane: \(da = ds d\phi\).

Gradient:
  \[ \vec{\nabla} f = \frac{\partial f}{\partial s} \hat{s} + \frac{1}{s} \frac{\partial f}{\partial \phi} \hat{\phi} + \frac{\partial f}{\partial z} \hat{z}. \]

Divergence:
  \[ \vec{\nabla} \cdot \vec{A} = \frac{1}{s} \frac{\partial}{\partial s} (s A_s) + \frac{1}{s} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}. \]

Curl:
  \[ \vec{\nabla} \times \vec{A} = \left( \frac{1}{s} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{s} + \left( \frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s} \right) \hat{\phi} + \frac{1}{s} \left( \frac{\partial (s A_\phi)}{\partial s} - \frac{\partial A_s}{\partial \phi} \right) \hat{z}. \]
1. The spherical shell of the figure carries a charge density $\rho(r) = \rho_0/r$, where $\rho_0$ is a constant, and $r$ is the radial coordinate. Using Gauss’s law:
   (a) [5 points] Find the expression of the electric field everywhere.
   (b) [2 points] Sketch a graph showing the electric field as a function of $r$.

2. The ring of the figure carries a uniform linear charge density $\lambda(s, \phi) = \lambda_0 \phi$, where $\lambda_0$ is a constant, and $\phi$ is the azimuthal angle.
   (a) [5 points] Calculate the electric potential on the $z$-axis.
   (b) [2 points] Calculate the energy necessary to bring a charge $q$ from infinite to the center of the ring (point $C$).
   (c) [2 points] Does the energy calculated in (b) depend on the path followed? Why?

3. [4 points] The electric field created by the infinite plane of the figure is:

   $$\vec{E} = \begin{cases} 
   E_0 \hat{z} & \text{if } z > 0 \\
   -E_0 \hat{z} & \text{if } z < 0 
   \end{cases}$$

   Using the boundary conditions of the electrostatic field, calculate the surface charge density carried by the plane.

4. On a certain region of the space there is an electric field $\vec{E} = e^{x^2} \hat{x} + \sin y \hat{y} + z^3 \hat{z}$.
   (a) [3 points] Which is the value of the associated charge density on the $xy$ plane?
   (b) [2 points] Is $\vec{E}$ a conservative field? Prove it.