Electrostatics (all source charges are stationary)

→ Coulomb's law and the principle of linear superposition form the basis for electrostatics

When 2 charges are stationary with respect to each other, then in the reference frame in which they are at rest, the force on \( q_1 \) due to \( q_2 \) is:

\[
\vec{F}_1 = \frac{q_1 q_2 \hat{R}}{4\pi\varepsilon_0 R^2}
\]

where

\[
\hat{R} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}
\]

\( \hat{R} \) is the unit vector from \( \vec{r}_2 \) to \( \vec{r}_1 \)

\( \vec{r} = \vec{r}_1 - \vec{r}_2 \) is the separation vector

and that on \( q_2 \) due to \( q_1 \) is

\[
\vec{F}_2 = -\vec{F}_1 = \frac{-q_1 q_2 \hat{R}}{4\pi\varepsilon_0 R^2}
\]

\( \varepsilon_0 \) is the electric permittivity of the free space [SI units]

\[
\varepsilon_0 = 8.85 \times 10^{-12} \frac{\text{F}}{\text{m}}
\]

\( F \rightarrow \text{Farad (capacitance)} \)

\( C \rightarrow \text{coulomb (charge)} \)

\( V \rightarrow \text{volt (electric potential)} \)

\( q_1, q_2 \); charges in C

\[
\begin{align*}
\varepsilon & = \frac{C}{V \cdot m} \\
\nu & = \frac{V}{c} \\
J & = \frac{c}{N \cdot \text{m}^2}
\end{align*}
\]
When a collection of charges, all at rest, are present then the force on the \( i \)-th charge is given by the linear superposition of the Coulomb forces on it due to each of the other charges:

\[
\vec{F}_i = \frac{q_i}{4\pi\varepsilon_0} \sum_{j\neq i} \frac{q_j}{r_{ij}^3} (\vec{r}_{ij})
\]

\[
= \frac{q_i}{4\pi\varepsilon_0} \sum_{j\neq i} \frac{q_j \hat{R}_{ij}}{R_{ij}} \quad \hat{R}_{ij} = \vec{r}_{ij} - \vec{r}_j 
\quad R_{ij} = |\vec{r}_{ij} - \vec{r}_j|
\]

"vector addition"

**Example:**

The principle of superposition seems "obvious" but what if the force were proportional to

\[q^2 \Rightarrow (q+q_2)^2 \neq q^2 + q_2^2 \]

**Diagram:**

\[
\vec{F}_a = \frac{q_1}{4\pi\varepsilon_0} \left( \frac{\hat{e}_1}{(a^2+\delta^2)} \right) + \frac{q_2}{4\pi\varepsilon_0} \left( \frac{\hat{e}_2}{(a^2+\delta^2)} \right)
\]

\[
\hat{e}_1 = \frac{a}{(a^2+\delta^2)^{1/2}} \hat{x} + \frac{\delta}{(a^2+\delta^2)^{1/2}} \hat{y} = \cos \theta \hat{x} + \sin \theta \hat{y}
\]

\[
\hat{e}_2 = \frac{a}{(a^2+\delta^2)^{1/2}} \hat{x} - \frac{\delta}{(a^2+\delta^2)^{1/2}} \hat{y} = \cos \theta \hat{x} - \sin \theta \hat{y}
\]

then:

\[
\vec{F}_a = \frac{q_1}{4\pi\varepsilon_0} \left( \frac{1}{(d^2+\delta^2)} \right) \left[ \cos \theta \hat{x} + \sin \theta \hat{y} + \cos \theta \hat{x} - \sin \theta \hat{y} \right]
\]

\[
\vec{F}_a = \frac{2q_1}{4\pi\varepsilon_0} \frac{\cos \theta \hat{x}}{d^2+a^2} = \frac{2q_1}{4\pi\varepsilon_0} \frac{a}{(d^2+a^2)^{3/2}} \hat{x}
\]
Electric Field

→ Provides a convenient description for electric force whenever arbitrary distributions of source charges contribute to the total electric force on a given ("test") charge

\[ \vec{F}_i = q_i \frac{1}{4\pi \varepsilon_0} \sum \frac{q_j \, \hat{R}_{ij}}{R_{ij}^2} = q_i \, \vec{E}_i(\vec{r}_i) \]

\[ \vec{E}_i = \frac{\vec{F}_i}{q_i} \]

i.e. the electric force on any charge may alternatively be expressed as the charge times the electric field, the latter given by the principle of superposition as

\[ \vec{E}(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \sum \frac{q_j \, \hat{R}_j}{R_j^2} \text{ where } \hat{R}_j = \vec{r} - \vec{r}_j \]

Note that by writing the electric field at a position \( \vec{r} \) rather than at the location of a charge \( \vec{r}_i \), we have given the \( \vec{E} \) field a continuum field character.

Note that while this was not entirely necessary here, this extension becomes crucial from a physical point of view. Indeed, both \( \vec{E} \) and the magnetic field, that we will discuss later, become physical entities that can carry energy, momentum, angular momentum and other attributes of a physical system.
due to a continuous charge distribution

If there are infinitely many point charges, each of vanishing magnitude in a region of space, so a certain finite amount of charge is contained in a unit volume of space, then one has the notion of a continuous charge distribution with a (volume) charge density, \( \rho \),

\[
\rho(\vec{r}') = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V}
\]

So the total charge in an infinitesimal volume \( dV \) at position \( \vec{r}' \) is

\[
dQ = \rho(\vec{r}') \, dV
\]

So

\[
\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_j \frac{Q_j \vec{r}_j}{R_j^3} \implies \frac{1}{4\pi\varepsilon_0} \int dV' \rho(\vec{r}') \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}
\]

unit vector

"Beware of vector superposition works with Cartesian components"

Sometimes we will use \( d\vec{r}' \) instead of \( dV' \).
Example (Problem 2.5 of the main text)

Find the electric field at a distance $h$ above the center of a circular loop of radius $a$ that carries a uniform line charge $\lambda$.

Note that the electric field of any element $d\vec{e}$ has components in the $xy$ plane that exactly cancel those from a corresponding element $-d\vec{e}$ in the diametrically opposite location on the loop. Therefore, the net field is going to be directed along the $z$-axis.

Since the inclination angle $\theta$ of the $(\vec{r}-\vec{r}')$ separation vector is the same for all elements $d\vec{e}$ and the distance $|\vec{r}-\vec{r}'|$ is also the same, the total field at distance $h$ along the $z$-axis is

$$\vec{E} = E_z \hat{z}$$

where

$$E_z = \frac{1}{4\pi \varepsilon_0} \frac{\cos \theta}{a^2 + h^2}$$

$E_z$ is the field due to a single element $d\vec{e}$.

The total charge $dq$ is

$$dq = \lambda \cdot d\vec{e}$$

and the radial distance $r$ is

$$r^2 = a^2 + h^2$$

The field due to the entire line charge is then

$$\vec{E} = \frac{\lambda a}{2\varepsilon_0} \frac{h}{(a^2 + h^2)^{3/2}} \hat{z}$$
Surface and line charge density

\[ \sigma(\vec{r}') : \text{surface charge per unit area at } \vec{r}' \text{ of a 2D surface} \]

\[ \lambda(\vec{r}'): \text{charge per unit length at } \vec{r}' \text{ on a linear charge distribution} \]

\[
\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma(\vec{r}') (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d\vec{r}' 
\]

\[
\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(\vec{r}') (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d\ell'
\]

Example (2.5) alternative cartesian-components approach

From the formula for \( \vec{E} \) for a line-charge distribution with \( \lambda(\vec{r}') = \lambda \) we have

\[
\vec{E}(\vec{r}) = \frac{\lambda}{4\pi\varepsilon_0} \int \frac{d\vec{r}'}{|\vec{r} - \vec{r}'|^3} \]

but as previously argued, the \( x \) and \( y \) components of the field must vanish on the \( z \)-axis due to symmetry.

Then:

\[
\vec{r}' = (0,0,h) \]

\[
1 - |\vec{r} - \vec{r}'|^2 = (x^2 + y^2 + h^2)^{3/2} = (a^2 + h^2)^{3/2}
\]

\( a^2 \) since \( (x',y') \) is on a circle of radius \( a \).

Then, we have for the non-zero \( z \) component of \( \vec{E} \)

\[
\vec{E}(\vec{r}) = \hat{z} E_z(0,0,h)
\]

\[
= \frac{\lambda}{4\pi\varepsilon_0} \int_0^{2\pi} \frac{h}{(a^2 + h^2)^{3/2}} \, d\phi' = \frac{\lambda h}{4\pi\varepsilon_0} \frac{1}{(a^2 + h^2)^{3/2}} \int_0^{2\pi} \frac{d\phi'}{(a^2 + h^2)^{3/2}}
\]

\[
= \frac{\lambda}{4\pi\varepsilon_0} \frac{h}{(a^2 + h^2)^{3/2}} \int_0^{2\pi} \frac{d\phi'}{(a^2 + h^2)^{3/2}} = \frac{\lambda}{2\pi\varepsilon_0} \frac{h}{(a^2 + h^2)^{3/2}} \]

\[ J = 38 \]
Notes
(1) Always decompose the vector integrand into its Cartesian components
(2) Integrate each Cartesian component of the vector integrand separately and then assemble results together for the final \( \vec{E} \) field.

Example (Problem 2.7 of the main text).
Find the electric field a distance \( z \) from the center of a spherical surface of radius \( R \) that carries a uniform charge density \( \sigma \). Treat the case \( z < R \) (inside) as well as \( z > R \) (outside). Express your answers in terms of the total charge \( q \) on the sphere.

Hint: Use the law of cosines to write \( r \) in terms of \( R \) and \( \theta \), be sure to take the positive square root \( \sqrt{R^2 + z^2 - 2Rz \cos \theta} \) if \( R > z \) but \( (R-z) \) if \( R < z \).

Again due to symmetry reasons, the electric field is directed along \( \hat{z} \) axis. The other components vanish.

\[
\vec{E}(\vec{r}) = \frac{\hat{z}}{4\pi \varepsilon_0} \sigma \int_{\text{sphere}} \frac{d\vec{a}'(z-z')}{1 - \vec{r} \cdot \vec{r}'} \quad \vec{r} = (0,0,h) \\
\vec{r}' = (x,y,z)
\]

\[
z - z' = h - z' = h - a \cos \theta' \\
|\vec{r} - \vec{r}'|^2 = (\vec{r} - \vec{r}') \cdot (\vec{r} - \vec{r}') = \\
= \hat{r} \cdot \hat{r}' + \hat{r}' \cdot \hat{r} - 2 \hat{r} \cdot \hat{r}' = \\
= h^2 + a^2 - 2ha \cos \theta'
\]

while \( d\vec{a}' = a^2 \sin \theta' d\theta' d\phi' \)
Then:

\[ \tilde{E}(\tilde{r}) = E_0 \sum_{\lambda} \frac{\mathcal{S}}{4\pi\epsilon_0} a^2 \int_0^{\pi} \sin \theta' \, d\phi' \int_0^{2\pi} \frac{h - a\cos \theta'}{(h^2 + a^2 - 2ah\cos \theta')^{3/2}} \, d\phi' \]

- Since the integrand has no \( \theta' \) dependence \( \int_0^{2\pi} d\phi' = 2\pi \)
- We can set \( \cos \theta' = \mu \) as new integration variable and therefore \( d\mu = -\sin \theta' \, d\theta' \)

So

\[ \tilde{E}(\tilde{r}) = E_0 \sum_{\lambda} \frac{\mathcal{S}}{2\pi\epsilon_0} a^2 \int_{-1}^{1} (-d\mu) \frac{h - a\mu}{(h^2 + a^2 - 2ah\mu)^{3/2}} \]

Notice that the integrand is a ratio of powers of quantities linear in \( \mu \), so can be easily integrated

Let \( h^2 + a^2 - 2ah\mu = u \) \( \Rightarrow \mu = \frac{h^2 + a^2 - u}{2ah} \) \( \Rightarrow d\mu = -\frac{du}{2ah} \)

Then

\[ I = \int_{-1}^{1} \frac{d\mu}{(h^2 + a^2 - 2ah\mu)^{3/2}} = \int_{-1}^{1} \frac{d\mu}{u^{3/2}} = \int_{-1}^{1} \frac{d\mu}{(h - a\mu)^{3/2}} \]

\[ = \frac{1}{2ah} \int_{-1}^{1} \frac{du}{(h - a\mu)^{3/2}} \frac{h - a^2/2h + u^{3/2}}{u^{3/2}} = \frac{1}{2ah} \int_{-1}^{1} \frac{du}{(h - a\mu)^{1/2}} \frac{h - a^2/2h + u^{3/2}}{u^{1/2}} \]

\[ = \frac{1}{2ah} \left[ \frac{h - a^2/2h}{-2u^{1/2}} \right]_{1-h/2}^{1+h/2} + \frac{1}{2ah} \left[ 2u^{1/2} \right]_{1-h/2}^{1+h/2} \]

\[ = \frac{1}{4ah^3} (h^2 - a^2) \left[ 1 - \frac{1}{1+h} \right] + \frac{1}{2ah^3} \left[ 1 - \frac{1}{h} \right] \]

\[ = \frac{1}{2ah^3} (h^2 - a^2) \left[ 1 - \frac{1}{1+h} \right] + \frac{1}{2ah^3} \left[ 1 - \frac{1}{h} \right] \]
For \( h > a \)
\[
\begin{align*}
1h-a1 &= h-a \\
1h+a1 &= h+a \\
I &= \frac{1}{2\pi h^2} \left[ \frac{(h+a)(h-a)}{h-a} - \frac{(h-a)(h-a)}{h+a} \right] + \frac{1}{2\pi h^2} \left[ h+a - h+a \right] \\
&= \frac{1}{2\pi h^2} \left[ h+a - h+a \right] + \frac{1}{2\pi h^2} \left[ h+a - a h \right] \\
&= \frac{1}{a h} + \frac{1}{a h} = \frac{2}{a h} \\
\text{then} \quad \overrightarrow{E}(\vec{r}) &= \hat{z} \frac{5}{2\pi \varepsilon_0} \frac{2a^2}{h^2} = \hat{z} \frac{5}{2\pi \varepsilon_0} \frac{a^2}{h^2} \quad \text{for} \ h > a
\end{align*}
\]

For \( h < a \)
\[
\begin{align*}
1h-a1 &= a-h \\
1h+a1 &= a+h \\
I &= \frac{1}{2\pi h^2} \left[ -\frac{(h-a)(a-h)}{a-h} + \frac{(h+a)(a-h)}{a+h} \right] + \frac{1}{2\pi h^2} \left[ h+a - (a-h) \right] \\
&= \frac{1}{2\pi h^2} \left[ -h-a + a-h \right] + \frac{1}{2\pi h^2} \left[ h+a - a + h \right] \\
&= \frac{-1}{a h} + \frac{1}{a h} = 0 \\
\text{Then:} \quad \overrightarrow{E}(\vec{r}) &= 0 \quad \text{for} \ h < a
\end{align*}
\]

We could have guessed this result using Gauss’s law
\[
\oint_S \overrightarrow{E} \cdot d\vec{a} = \frac{Q_{\text{enc}}}{\varepsilon_0}
\]

Since \( Q_{\text{enc}} = 0 \) for \( h < a \) \( \Rightarrow \overrightarrow{E} = 0 \)