Effect of $\mathbf{B}$ on a circularly orbiting charge
(as in an "atom") — Diamagnetism

A classical atom is one in which an electron is in a circular orbit around a fixed nucleus that provides the needed centripetal force:

$$\frac{Me}{r} \cdot \frac{V^2}{r} = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r^2}$$

The magnetic dipole moment of the electronic orbit is given by $\vec{m} = -\vec{e} \cdot \left( \frac{V}{2\pi r} \right) \pi r^2 = -\vec{e} \cdot \frac{evr}{2}$

Now place a $\mathbf{B}$ field perpendicular to the orbit, i.e. along $\hat{z}$. That provides an extra centripetal force inward $-evB\hat{z}$ which changes the speed of the electron to $v+\Delta v$:

$$\frac{Me}{r} \cdot (v+\Delta v)^2 = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r^2} + evB$$

Taking this correction to be small and the radius of the orbit to be unperturbed (which can be shown to be true), we have
\[
\text{Me} \left( \frac{v^2 + 2v\Delta v + O(\Delta v^2)}{r} \right) = \frac{1}{q\pi\varepsilon_0} \frac{e^2}{r^2} + e\nu B
\]

or

\[
\frac{\text{Me}v^2}{r} + \frac{2\text{Me}v\Delta v}{r} = \frac{1}{q\pi\varepsilon_0} \frac{e^2}{r^2} + e\nu B
\]

\[
\Rightarrow \Delta v = \frac{eBr}{2me}
\]

Thus, \( \Delta \bar{m} = \text{change in magnetic dipole moment of electron orbit} \)

\[
= -\frac{\hat{z} \cdot e}{2} \Delta v = -\frac{3}{2} \frac{e^2 Br^2}{4me} = -\frac{e^2}{4me} \frac{r^2}{B}
\]

which is oppositely directed to \( \vec{B} \). Thus as \( \vec{B} \) increases, \( \bar{m} \) decreases, quite opposite of what happens for electron spin. This is diamagnetism. With proper interpretation this formula can be shown to be valid quantum mechanically as well, at least to factors of order 1.

Comment 1: If the speed of the orbiting electron is larger when \( \vec{B} \neq 0 \), then, since the radius of the orbit is unchanged, the total energy, \( \frac{1}{2} \text{mv}^2 + \frac{e^2}{r} \) (\( t=1 \) for H-atoms), is large too. But the magnetic field does no work, \( -e\nu \times \vec{B} \perp \nu \), but then where is the extra energy coming from?
The answer to this question is that no matter how slowly $\vec{B}$ is changed from its initial value to its final nonzero value, such changing $\vec{B}$ fields induce an electric field via the Faraday effect, which we have not discussed since that is outside magnetostatics. It is this $\vec{E}$ field that does work and increases the kinetic energy of the electron. Indeed, without such induced $\vec{E}$ fields, one could create arbitrarily strong $\vec{B}$ fields without expending any energy at all in violation of what we know is not true.

Comment 2:
Why is $r$ fixed while $\vec{B}$ is being applied?

The answer to this question is related to that of the previous question. If one allows for the change of both $r$ and $\vec{v}$, then one can show that to linear order in $\vec{B}$ and in $\Delta r, \Delta v$

$\Delta r = 0$ identically!

Thus if the charge were being rotated tied to the end of a stretchable spring, then as the $\vec{B}$ field is applied to the system the accelerating motion of the charge along the azimuthal direction would leave the spring extension unchanged. The increased centripetal force required to rotate the charge at increasing angular speed is furnished exactly by the $\text{e}v\times\vec{B}$ force of the magnetic field.
Notice: To show this, one needs to use Faraday's law to determine the induced azimuthally-directed $\vec{E}$ field induced by the changing magnetic field and use the work-energy theorem to show that

$$\Delta V = \frac{e \mathbf{r}}{2\pi \varepsilon_0} \mathbf{A} \mathbf{B}$$

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### Magnetized objects and Magnetization

A magnetized object can be described by specifying the average magnetic dipole moment per unit volume, much as an electrically polarized object. This is magnetization density, or simply magnetization, denoted as $\mathbf{M}(\mathbf{r})$

The vector potential $\mathbf{A}$ for such an object is given by integrating over the contributions from individual atomic magnetic dipoles

$$\mathbf{A}_M(\mathbf{r}) = \frac{\mu_0}{4\pi} \mathbf{r} \times \mathbf{M}(\mathbf{r}) \longrightarrow \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}) \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d\mathbf{r}'$$

i.e. $\mathbf{A}_M(\mathbf{r}) = \frac{\mu_0}{4\pi} \left\{ \mathbf{r} \times \left[ \mathbf{M}(\mathbf{r}) \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \right) - \mathbf{r}' \times \left( \frac{\mathbf{M}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|^3} \right) \right] + \mathbf{r} \times \mathbf{M}(\mathbf{r}) \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \right\}$

Used:

$$\nabla \cdot \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\frac{1}{|\mathbf{r} - \mathbf{r}'|^3}$$

$$\nabla \times \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

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\[ \vec{A}_m(\vec{r}) = -\frac{\mu_0}{4\pi} \oint_V \frac{\vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\vec{r}' + \frac{\mu_0}{4\pi} \oint_V \frac{\vec{\nabla} \times \vec{M}(\vec{r})}{|\vec{r} - \vec{r}'|} \, d\vec{r}' \]

from a variant of the divergence theorem.

\[
\oint_S \vec{M} \cdot d\vec{n} = \oint_V \frac{\vec{\nabla} \times \vec{M}(\vec{r})}{|\vec{r} - \vec{r}'|} \, d\vec{r}'
\]

So \( \vec{A}_m(\vec{r}) = \frac{\mu_0}{4\pi} \oint_S \frac{\vec{M}(\vec{r}') \times \vec{n}}{|\vec{r} - \vec{r}'|} \, d\vec{a}' + \frac{\mu_0}{4\pi} \oint_V \frac{\vec{\nabla} \times \vec{M}(\vec{r})}{|\vec{r} - \vec{r}'|} \, d\vec{r}' \)

The second term begs the analogy of \( \vec{\nabla} \times \vec{M}(\vec{r}') \) with a form of current, since for a true current \( \vec{J}(\vec{r}') \)

\[ \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times \vec{n}}{|\vec{r} - \vec{r}'|} \, d\tau' \]

Indeed, \( \vec{\nabla} \times \vec{M}(\vec{r}) \) may be interpreted as the contribution of bound currents, while \( \vec{J}(\vec{r}') \) denotes currents carried by macroscopically free charges, or free currents. This interpretation is quite apt, since magnetization arises from the spins and orbital motion of electrons (and other charges and particles), representing localized bound currents.

Thus \( \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \) must be modified in the presence of magnetic materials to

\[ \vec{\nabla} \times \vec{B} = \mu_0 \left[ \vec{J}_f + \vec{\nabla} \times \vec{M} \right] \]

i.e., \( \vec{\nabla} \times (\vec{B} - \mu_0 \vec{M}) = \mu_0 \vec{J}_f \)

Defining \( \vec{A} = \frac{1}{\mu_0} \vec{B} - \vec{M} \) as a new auxiliary field, much as \( \vec{D} \) was in electrostatics.
\[ \vec{\nabla} \times \vec{H} = \vec{J}_f \]
only free current density serving as the source for the \( \vec{H} \) field. This field is sometimes known as the magnetic field intensity, while \( \vec{B} \) is called magnetic field induction, but we will avoid this confusion, outdated nomenclature and call \( \vec{H} \) simply \( \vec{H} \) field.

⇒ Similarly the first term

\[ \frac{\mu_0}{4\pi} \oint_S \frac{\vec{M}(\vec{r}')}{1 - \vec{r} - \vec{r}'/l} \, d\vec{a'} \]

corresponds to the vector potential of a surface current density \( \vec{K}_m = \vec{M} \times \hat{n} \).

In summary:

\[ \vec{A}_m(\vec{r}) = \frac{\mu_0}{4\pi} \oint_S \frac{\vec{M}(\vec{r}') \times \hat{n}}{1 - \vec{r} - \vec{r}'/l} \, d\vec{a}' + \frac{\mu_0}{4\pi} \int_V \frac{\vec{\nabla}' \times \vec{M}(\vec{r}')}{1 - \vec{r} - \vec{r}'/l} \, d\vec{r}' \]

Physical interpretation

• If \( \vec{M} \) is uniform throughout the magnetized body, then \( \vec{J}_m = 0 \) but \( \vec{K}_m \neq 0 \).

Note that all internal bound currents cancel out leaving only the uncanceled current on the surface of amount \( \vec{m} \) per unit width.

\[ \vec{K}_{d\vec{t}} = \frac{d\vec{m}}{d\vec{x} d\vec{y}} = \frac{(\vec{M} \times d\vec{z} \, dx \, dy)}{\text{area of each tile}} \rightarrow \text{magnetic dipole moment of each tile} \]
• For nonuniform \( M \), let \( M \) be once again along \( \hat{z} \)
  but its magnitude vary with \( y \) only in the flat slab of form shown on the previous page, i.e. \( M = M(y) \hat{z} \)

\[
\nabla \times \left( M(y) \hat{z} \right) = \frac{\partial M}{\partial y} \hat{x} \times \hat{z} = \frac{\partial M}{\partial y} \hat{x} \times \hat{z}
\]

\[
= \frac{\partial M}{\partial y} \hat{x} \times \hat{z}
\]

\[
\Rightarrow J_x (dy \, dz) = \frac{\partial M}{\partial y} dy \, dz
\]

cross-sectional area to current flow.

i.e. \( J_x = \frac{\partial M}{\partial y} \), no other component to the current flow

i.e. \( \bar{J} = x \frac{\partial M}{\partial y} \hat{e} \times (M(y) \hat{e}) = \vec{\nabla} \times (M(y) \hat{e}) = \vec{\nabla} M(y) \times \hat{e} = \frac{\partial M}{\partial y} \hat{x} \times \hat{e} = \frac{\partial M}{\partial y} \hat{x} \times (\hat{e} \times \hat{e}) = \frac{\partial M}{\partial y} \hat{x} \times \hat{e} \)

\( \checkmark \) checks

• When \( M \) is arbitrary, \( \bar{J}_m = \vec{\nabla} \times \bar{M} \) gives the volume current density of magnetization related bound current density