Magnetic dipole moment of point charge in motion

Let a point charge of amount $q$ and mass $m_q$ be in motion at velocity $\vec{V}_q$. The current density contributed by this charge is

$$\vec{J} = q \, \delta^{(3)}(\vec{r} - \vec{r}_q) \, \vec{V}_q$$

$\rho(\vec{r})$, volume charge density

Now, the magnetic dipole moment

$$\vec{m} = I \int d\vec{a} = I \frac{1}{2} \oint_{\Sigma} \vec{r} \times d\vec{e}$$

Since

$$|\vec{r} \times d\vec{e}| = |\vec{r}||d\vec{e}| \sin \Theta = \text{area of parallelogram formed by } \vec{r} \text{ and } d\vec{e}$$

$$= 2 \times \text{area of shaded triangle}$$

and $\vec{r} \times d\vec{e}$ is oriented $\perp$ to the plane formed by $\vec{r}$ and $d\vec{e}$

$$\Rightarrow \frac{1}{2} \vec{r} \times d\vec{e} = d\vec{a} \text{ for the shaded triangle}$$

Thus

$$\vec{m} = \frac{1}{2} \int \vec{r} \times d\vec{e}$$

$$= \frac{1}{2} \int \vec{r} \times J \, d\tau$$

$\tau$ for volume currents

For the above then

$$\vec{m} = \frac{q}{2} \int \vec{r} \times \vec{V}_q \delta(\vec{r} - \vec{r}_q) \, d\tau = \frac{q}{2} \vec{V}_q \times \vec{V}_q = \frac{q}{2m_q} \vec{V}_q \times (m_q \vec{V}_q)$$
Therefore

\[ \vec{m} = \frac{q}{2m_q} \vec{v}_q \times (m_q \vec{v}_q) \]

\[ \vec{L}_q \rightarrow \text{angular momentum} \]

\[ \vec{m} = \frac{q}{2m_q} \vec{L}_q \]

i.e. the magnetic dipole moment of a moving charge is proportional to its angular momentum.

A similar result holds for the intrinsic spin angular momentum as we will see later, which must be added for total \( \vec{m} \).

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Magnetic Materials

Just as dielectrics respond to an electric field by means of an induced polarization, certain media when exposed to an external magnetic field respond to it by means of induced magnetic dipole moment, or what is related to magnetization density. Since a magnetic dipole moment is a property of a current in a closed loop, magnetic media respond to a magnetic field in terms of perturbations of bound atomic currents (of course, a conductor has both electric and magnetic responses due to the fact that macroscopic free charges can cause charge separation and currents on macroscopic scales, but when discussing magnetic media one often restricts attention to bound magnetic currents alone).
Unlike dielectrics in which $\tilde{\mathbf{p}}$ is often along $\tilde{\mathbf{E}}$ (except in anisotropic, crystalline materials), in nature a significant fraction of magnetic materials exist for which $\tilde{\mathbf{m}}$ & $\tilde{\mathbf{B}}$ are antiparallel. Such materials are known as diamagnetic and are distinguished from materials known as paramagnetic which are the best analog of linear dielectrics in which $\tilde{\mathbf{m}}$ & $\tilde{\mathbf{B}}$ are parallel. Yet another class of magnetic materials, called ferromagnetic, exist in which the magnetic dipole moment is present even when no external magnetic field is. Bar magnets are made of such materials that contain aligned magnetic moments frozen in large domains created by large magnetic fields during the magnetization process but which are unable to relax when these fields are taken away. Another complication not present in dielectrics has to do with the fact that elementary particles like electrons, protons and neutrons that make up atoms and molecules carry intrinsic spins that have a purely quantum mechanical origin but yield magnetic dipole moments.
The strongest magnetic signature of a medium has in general to do with such intrinsic spin-dependent magnetic moments of the electrons of form
\[ \vec{m}_e = -g \frac{e}{2\hbar} \vec{S} \]
\( \vec{S} = (S_x, S_y, S_z) \) is the spin-angular momentum vector
\(-e = \text{electron charge}\)
\( g = 2, \) gyromagnetic ratio for the electron.

An atom containing an even \# of paired electrons in its outer shell has no such magnetic moment since the electron spin vectors are paired in opposite directions ("up-down" pairing.) In such cases, only the orbital motions of the electrons yield any magnetic dipole moment, and their perturbation by an external \( \vec{B} \) field causes a net magnetic dipole moment that is opposite to the \( \vec{B} \) field, resulting in diamagnetic behavior. But when the outer shell of an atom has an odd \# of electrons, such an atom is guaranteed to be paramagnetic, since there will be at least one (possibly more) unpaired electron spins. In such cases, the perturbation of orbital motion of the atomic electrons is a much weaker effect and can be ignored.
Torques & forces on magnetic dipoles and paramagnetics

In the presence of an uniform magnetic field $\mathbf{B}$, a magnetic dipole moment $\mathbf{m}$ experiences a torque. To see this, consider the torque on a circular loop current in the xy-plane:

$$
\mathbf{N} = \int \mathbf{r} \times d\mathbf{F} = \int \mathbf{r} \times (I d\mathbf{e} \times \mathbf{B})
$$

$$
= IR^2 \int_0^{2\pi} d\phi \mathbf{r} \times (\mathbf{r} \times \mathbf{B})
$$

$$
= IR^2 \int_0^{2\pi} d\phi \left[ (\mathbf{r} \cdot \mathbf{B}) \mathbf{r} - (\mathbf{r} / r_0) \mathbf{B} \right]
$$

$$
= IR^2 \int_0^{2\pi} d\phi \left( \cos \phi B_x + \sin \phi B_y \right) (-\sin \phi \mathbf{x} + \cos \phi \mathbf{y})
$$

$$
= IR^2 \int_0^{2\pi} d\phi \left( \cos^2 \phi B_x \mathbf{y} - \sin^2 \phi B_y \mathbf{x} \right)
$$

But

$$\int_0^{2\pi} \cos \phi d\phi = \frac{1}{2} \int_0^{2\pi} (1 - \cos 2\phi) d\phi = \frac{1}{2} \cdot 2\pi = \pi$$

$$\int_0^{2\pi} \sin \phi d\phi = \pi$$

$$\Rightarrow \mathbf{N} = (\pi R^2) I (B_x \mathbf{y} - B_y \mathbf{x}) = I (\pi R^2) \mathbf{r} \times (B_x \mathbf{x} + B_y \mathbf{y} + B_z \mathbf{z})$$

$m$, magnetic dipole moment of current loop

$\Rightarrow \mathbf{N} = \mathbf{m} \times \mathbf{B}$
Note: The text evaluates $\mathbf{N}$ for a square current loop. One can show that in general for any arbitrary current loop of any size, $\mathbf{N} = \mathbf{m} \times \mathbf{B}$, if the field $\mathbf{B}$ is uniform. Indeed, this expression holds for any localized current distribution in the presence of a uniform field. Also note that the force $\mathbf{F}$ on a current loop in a uniform $\mathbf{B}$ field vanishes since:

$$\mathbf{F} = I \oint \mathbf{d} \mathbf{e} \times \mathbf{B} = I \left( \oint \mathbf{d} \mathbf{e} \right) \times \mathbf{B}$$

$$= 0 \text{ since } \oint \mathbf{d} \mathbf{e} = 0$$

As a result, $\mathbf{N}$ is independent of the choice of the origin.

The torque on a magnetic dipole $\mathbf{m}$ tends to align it along the external $\mathbf{B}$ field, which is the principal mechanism for paramagnetism. (Question: why does $\mathbf{m}$ not tend to anti-align relative to $\mathbf{B}$, for which $\mathbf{N} = \mathbf{m} \times \mathbf{B}$ is also $0$? Answer: anti-alignment yields an unstable equilibrium and is thus not observed experimentally in general.)
Force on a magnetic dipole in a nonuniform $\bar{B}$ field

Just like a permanent $\bar{p}$ dipole in an external nonuniform $\bar{E}$ field for which

$$\vec{F} = \nabla (\vec{p} \cdot \vec{E})$$

a permanent magnetic dipole, or current loop, experiences a force in a nonuniform $\bar{B}$ field of amount

$$\vec{F} = \nabla (\vec{m} \cdot \vec{B})$$

very analogously.

This can be shown for a planar rectangular loop rather simply.

$$\vec{F} = \int \int \int \vec{B}(\vec{r}') \times (\vec{r}' \cdot \nabla) \vec{B}_0 \, \mathrm{d} \vec{r}'$$

Write $\vec{B}(\vec{r}') = \vec{B}_0 + (\vec{r}' \cdot \nabla) \vec{B}_0 \big|_0 + \ldots$

so

$$\vec{F} = \int \int \int \vec{B}_0 \times \left[ (\vec{r}' \cdot \nabla) \vec{B}_0 \right] \, \mathrm{d} \vec{r}'$$

but $\oint \vec{B} \, \mathrm{d} \vec{l} = 0$ so the first term does not contribute.

$$\vec{F} = \int \int \int \vec{B}_0 \times (\vec{r}' \cdot \nabla) \vec{B}_0 \, \mathrm{d} \vec{r}'$$

Now $\vec{d} \vec{r}' = \mathrm{d} x' \hat{x} + \mathrm{d} y' \hat{y}$

so

$$\vec{F} = \int \int \int \vec{B}_0 \times \left[ \hat{z} \times \nabla \vec{B}_0 \right] \, \mathrm{d} \vec{r}'$$

$$= \int \int \left\{ \hat{z} \times \left[ \nabla \int \vec{B}_0 \, \mathrm{d} \vec{r}' \right] \right\}$$

but $\int \vec{d} (\frac{1}{2} x'^2) = 0$ as is $\int y' \, \mathrm{d} y'$
\[ F = \int x' dy' \left( \frac{\partial}{\partial y} \right) - \int y' dx' \left( \frac{\partial}{\partial x} \right) \]

Now, \( \int x' dy' = \int x dy \) since on II and IV, \( dy' = 0 \) and on III \( x' = 0 \)

\[ \Rightarrow \int x' dy' = \text{area of the loop} \]

Similarly, \( \int y' dx' = \int y dx \)

\[ \Rightarrow F = Ia \left[ \frac{\partial}{\partial y} (y x \vec{B}) \right] - \frac{\partial}{\partial x} (x y \vec{B}) \]

\[ = Ia \left( \frac{\partial}{\partial y} (\vec{V} x \vec{B}) \right) = (\vec{m} x \vec{V}) x \vec{B} \]

but \( (\vec{m} x \vec{V}) x \vec{B} = \vec{V} (\vec{m} \cdot \vec{B}) - m (\vec{V} \cdot \vec{B}) \)

\[ \Rightarrow F = \vec{V} (\vec{m} \cdot \vec{B}) \]

Note: although derived for a rectangular loop, the above is valid for any arbitrary shape planar loop since \( \int y dx = \int x dy = A \) for any planar loop in the xy plane.

Indeed, since any localized current distribution may be composed from a sufficient number of sufficiently small planar loops, this result is valid for any such current distribution. The only requirement is that the spatial extent of this distribution be small when compared to the scale over which \( \vec{B} \) varies so the 1st order Taylor expansion for \( \vec{B} \) is a valid, accurate approximation.