Examples of Ampere's Law

1. Infinitely long, straight current-carrying wire

- From azimuthal symmetry and the azimuthal orientation of \( \mathbf{B} \)

\[
\mathbf{B}(r) = B(r) \hat{\phi}
\]

Then we have for an Amperian loop of radius \( s \) centered at the wire

\[
\oint \mathbf{B} \cdot d\mathbf{e} = B 2\pi s = \mu_0 I
\]

\[\Rightarrow B = \frac{\mu_0 I}{2\pi s}\]

i.e.

\[
\mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}
\]

2. A plane surface current \( \mathbf{K} \) that is uniform everywhere

From the right hand rule, \( \mathbf{B} \) field is as shown above the current plane and oppositely directed below.
\[ B \cdot E + E \cdot B = \mu_0 \kappa \cdot E \]

\[ B = \frac{\mu_0 \kappa}{2} \quad \text{above or below} \]

\[ \vec{B} = \frac{\mu_0}{2} \vec{k} \times \hat{n} \]

\( i.e. \) when direction is accounted for \( \hat{n} \) being the outward normal

\text{Note that}

\[ \vec{B}_+ - \vec{B}_- = \frac{\mu_0}{2} \vec{k} \times \hat{n} - \frac{\mu_0}{2} \vec{k} \times (-\hat{n}) \]

\[ = \mu_0 \vec{k} \times \hat{n} \]

\( i.e. \) the \( \vec{B} \) field is tangential and discontinuous by amount \( \mu_0 \vec{k} \times \hat{n} \) across a surface current.

\text{Cf.} \vec{E} \text{ is discontinuous across a surface charge by amount} \ \frac{\sigma}{\varepsilon_0} \hat{n}.

\text{A direct evaluation - from Biot-Savart, since} \ \vec{k} \text{ is uniform}

\[ \vec{B} = \frac{\mu_0}{4\pi} \vec{k} \times \left( \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right) \, da' = \frac{\mu_0}{4\pi} \vec{k} \times \frac{\sigma}{2\varepsilon_0} \hat{n} \]

\[ \vec{E} \text{ field due to a uniform charge density} \]

\[ \sigma = 4\pi \varepsilon_0 \]

\[ = \frac{\mu_0}{4\pi} \frac{4\pi \varepsilon_0}{2\varepsilon_0} \vec{k} \times \hat{n} = \frac{\mu_0}{2} \vec{k} \times \hat{n} \quad \text{as derived above.} \]
(3) Infinitely long solenoid of circular cross-section with \( n \) turns per unit of length, current \( I \) in each turn.

From symmetry and the right-hand rule, the \( \vec{B} \) field can only be longitudinal, i.e., parallel to the axis of the solenoid, everywhere.

To see this note that if \( \vec{B} \) had an azimuthal component \( B_\phi \), then by means of an Amperean loop that is concentric with the solenoid, we see that

\[
\oint \vec{B} \cdot d\vec{e} = B_\phi 2\pi s = 0 \quad \text{no current enclosed}
\]

\[ \Rightarrow \boxed{B_\phi = 0} \]

To see why \( B_\phi = 0 \) too, note that if \( I \rightarrow -I \) i.e. the direction of the current is reversed, then \( \vec{B} \rightarrow -\vec{B} \), i.e., \( B_\phi \rightarrow -B_\phi \) (from Biot-Savart law).

But reversing the current is equivalent to flipping the solenoid left to right, or looking at it from behind it, which cannot make any difference to \( B_\phi \Rightarrow B_\phi \rightarrow B_\phi \). This is only consistent with \( B_\phi \rightarrow -B_\phi \) if \[ \boxed{B_\phi = 0} \]
Thus, $\mathbf{B} = Bz$ everywhere. Further, both inside and outside the solenoid, $B_z$ must be independent of distance from the axis, as a simple application of Ampere's law on the rectangular loop shown below.

\[
[\mathbf{B}_z(s_1) - \mathbf{B}_z(s_2)] \cdot \mathbf{i} = 0 \Rightarrow \mathbf{B}_z(s_1) = \mathbf{B}_z(s_2)
\]

Since as $s \to \infty$, $\mathbf{B}(s) \to 0$, this also implies that $\mathbf{B} = 0$ everywhere outside the solenoid.

With all these observations together, we conclude that the field $\mathbf{B}$ is everywhere longitudinal, uniform and non-zero inside, but 0 everywhere outside. Note that this conclusion is only valid if the solenoid is infinitely tightly wound. For practical windings, as shown below, the $\mathbf{B}$ field has the following form.

We may now use Ampere's law to find the $\mathbf{B}$ field for an idealized solenoid.
The Amperian loop shown is partially inside and partially outside.

\[ \oint \mathbf{B} \cdot d\mathbf{e} = B \cdot e + 0 + 0 + 0 = \mu_0(n I) \text{I} \]

\[ \text{total current enclosed} \]

\[ \# \text{turns/length} \]

\[ \Rightarrow B = \mu_0 n I \]

\[ \begin{bmatrix} \mathbf{B} = \mu_0 n I \hat{I} \end{bmatrix} \text{everywhere inside} \]

\[ \text{and} \ 0 \ \text{outside.} \]

**Note**: The circular nature of the cross-section is entirely secondary, insofar as any infinite solenoid with a cross-section that does not change along its length produces a uniform axial B field everywhere inside and a zero B field outside in the idealized sense. This can be argued by means of the Biot-Savart law.

(4) A toroidal solenoid with tightly but uniformly wound turns, carrying current I and with N turns in all
the cross-sectional shape can be arbitrary as long as the equivalent points in the cross-sections at different positions around the toroid lie on circles. This requires the cross-sectional geometry (shape, size, orientation) to be the same around the toroid.

for such toroids, the $B$ field is circumferential inside, but zero outside

\[ \oint B \cdot dl = \mu_0 NI \]

$B = \mu_0 NI$ \( \phi \) inside \[ \frac{\mu_0 NI}{2\pi s} \phi \) outside

Magnetic vs Electric forces

Consider two wires, carrying the same current $I$ placed a distance $s$ apart. If the neutralizing positive ions were completely removed from the two wires, we would have two fully electrically charged wires with a charge density $\lambda$ and current $I = I V$, where $V$ is the speed of the moving charges (electrons). Then, both electric and magnetic forces are present. Let us now compare them.
Since \( \vec{E} = \frac{A}{2\pi\varepsilon_0} \frac{\hat{s}}{s} \) (radial)

Electric on wire 2 (due to wire 1)

\( \vec{E}_{\text{electric}} = \Lambda \hat{E} \) (per unit of length)

\[ \frac{A^2}{2\pi\varepsilon_0} \frac{\hat{s}}{s} \] (repulsive)

Whereas

\[ \vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi} \] (azimuthal) so

Magnetic on wire 2 (due to wire 1)

\[ \vec{F}_{\text{mag}} = I \hat{z} \times \vec{B} \] (per unit of length)

\[ = -\frac{\mu_0 I^2}{2\pi s} \hat{s} \] (attractive)

\[ = -\frac{\mu_0 I^2 V^2}{2\pi s} \hat{s} \]

Then

\[ \frac{F_{\text{mag}}}{F_{\text{electric}}} = \frac{\mu_0 \varepsilon_0 V^2}{c} = \frac{V^2}{c^2} \] \( (\mu_0 \varepsilon_0 = \frac{1}{c^2}) \)

But \( v \approx \frac{\text{wavelength}}{s} \Rightarrow \frac{v}{c} \approx 10^{-11} \Rightarrow \frac{F_{\text{mag}}}{F_{\text{electric}}} \approx 10^{-22} \)

\[ \text{very tiny} \]

see prob. 5.19
from the text