Multipole expansion of the potential of a localized charge distribution

Consider a localized charge distribution characterized by volume charge density \( \rho(r') \) and the potential generated by it at some observation point \( \vec{r} \). Clearly, if \( \vec{r} \) is very far away from the distribution, it must resemble a point charge from the perspective of the point \( \vec{r} \), and \( V(\vec{r}) \) must have the \( \frac{Q}{4\pi\epsilon_0 r} \) form corresponding to it, where \( Q \) is the total charge of the distribution.

But, if \( Q \) happens to be 0, i.e., the distribution on the whole is neutral, then what does the potential of it look like at the far away point?

This is the purpose of the multipole expansion, to show that the potential may be expressed as a succession of "higher-order" terms, each with a progressively more rapid power-law decay at large distances. We will see in this process the emergence of the dipole moment, quadrupole moment, octupole moment, and so on of the charge distribution.
From the Coulomb law:

\[ V(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r')}{|r-r'|} dr' \]

Now

\[ \frac{1}{|r-r'|} = \frac{1}{\sqrt{(r-r')(\mathbf{r}-\mathbf{r}')}} = \frac{1}{\sqrt{r^2 - 2rr'\cos\delta + r'^2}} \]

or

\[ \frac{1}{\sqrt{r^2 - 2rr'\cos\delta + r'^2}} = \frac{1}{\sqrt{r^2 - 2rr'\cos\delta + r'^2}} \]

where

\[ \mathbf{r}\cdot\mathbf{r}' = rr'\cos\delta \]

is the angle between \( \mathbf{r} \) and \( \mathbf{r}' \).

For \( r>r' \), one may expand this expression into a power series of powers of \( \frac{r'}{r} \). It turns out that such a power series may be written down exactly in terms of the Legendre polynomials of \( \cos\delta \) as

\[ \frac{1}{|r-r'|} = \frac{1}{\sqrt{r^2 - 2rr'\cos\delta + r'^2}} \]

\[ = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos\delta) \quad \text{for } r>r' \]

Also Known as the generating function for the \( P_l \)s.

Thus, on substitution into the expression for \( V \), we have:

\[ V(r) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int r'^l \rho(r') P_l(\cos\delta) dr' \]

Although this expression appears quite dense, its low-order terms may be easily simplified.
Thus, writing
\[ V(\vec{r}) = \sum_{\ell=0}^{\infty} V_{\ell}(\vec{r}) \]

where \( V_{\ell}(\vec{r}) \): contribution of the \( \ell \) term
\[
= \frac{1}{4\pi\varepsilon_0} \frac{1}{r^{\ell+1}} \int r' P_{\ell}(\cos \theta') P_{\ell}(\cos \theta) \, d\Omega'
\]

we have:
\[
\begin{align*}
\boxed{\ell=0} & \quad V_0(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \int \left. \frac{\rho(\vec{r}')}{\cos \theta} \right|_{\theta=0} \, d\Omega' = \frac{Q_{\text{tot}}}{4\pi\varepsilon_0} \frac{1}{r} \\
& = \boxed{Q_{\text{tot}} = \int \rho(\vec{r}') \, d\Omega'}
\end{align*}
\]

\[
\begin{align*}
\boxed{\ell\geq 1} & \quad V_{\ell}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^{\ell+2}} \int r' P_{\ell}(\cos \theta') P_{\ell}(\cos \theta) \, d\Omega'
\end{align*}
\]

but
\[
r' \cos \theta' = \frac{r' \cos \theta}{r} = \frac{\vec{r}' \cdot \vec{r}}{r}
\]

so
\[
V_{\ell}(\vec{r}) = \frac{\vec{P} \cdot \vec{r}}{4\pi\varepsilon_0 r^3} \cdot \int \frac{\vec{r}' P_{\ell}(\cos \theta')}{\cos \theta} \, d\Omega'
\]

\[
= \boxed{\vec{P} = \int \vec{r}' P_{\ell}(\cos \theta') \, d\Omega'}
\]

\[
\text{contribution of the electric dipole to the potential.}
\]
Example of an electric dipole

\[ \mathbf{p} = \int \mathbf{r}' \delta(\mathbf{r}' - \mathbf{r}) \, d\mathbf{r}' \]

\[ = \int \mathbf{r}' \left[ q \delta^{(3)}(\mathbf{r}' - \mathbf{q}) - q \delta^{(3)}(\mathbf{r}' - \mathbf{r}_q) \right] \, d\mathbf{r}' \]

\[ = q (\mathbf{r}_q - \mathbf{r}_q) = q (\mathbf{r}_q - \mathbf{r}_q) = q \mathbf{d} \]

So, a pair of equal and oppositely signed charges, \( \pm q \), when separated, form an electric dipole of moment \( q \mathbf{d} \), \( \mathbf{d} \) being the separation vector directed from the negative to the positive charge.

A point dipole — the limiting case \( q \to \infty \), \( d \to 0 \), so that \( \mathbf{p} = q \mathbf{d} \) is finite is called a point dipole.

More about the \( l=1 \) (dipole) term

How about the electric field in the \( l=1 \) (dipole) order?

\[ \mathbf{E}_1(\mathbf{r}) = - \nabla \mathbf{V}_1(\mathbf{r}) \]

\[ = - \nabla \left( \frac{\mathbf{p} \cdot \mathbf{r}}{4 \pi \varepsilon_0 r^3} \right) = - \frac{1}{4 \pi \varepsilon_0} \nabla \left( \sum_{i=1}^{3} \frac{P_i r_i}{r^3} \right) \]

\[ i=1 \to \nabla \left( \frac{x}{r^3} \right) = \nabla (x) \frac{1}{r^3} + x \nabla \left( \frac{1}{r^3} \right) = \frac{x}{r^3} - 3 x \frac{r^2}{r^5} \]

and similarly for \( i=2,3 \).
Thus
\[
\vec{E}_i(\vec{r}) = -\frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{3} \vec{P}_i \left[ \hat{r}_i \frac{1}{r^2} - \frac{3 \hat{r}_i \cdot \hat{r}}{r^5} \right]
\]
\[
= \frac{1}{4\pi\varepsilon_0} \left[ \frac{3 \vec{P} \cdot \hat{r}}{r^5} \left( \frac{2 P_i r_i}{\hat{r} \cdot \hat{r}_i} \right) - \frac{1}{r^2} \sum_{i=1}^{3} P_i \hat{r}_i \right]
\]
\[
= \frac{1}{4\pi\varepsilon_0} \left[ \frac{3 (\vec{P} \cdot \vec{r}) \hat{r}}{r^5} - \frac{\vec{P}}{r^3} \right]
\]

**Example:** \( \vec{P} = q \hat{d} \), along \( \hat{z} \)

\[
\vec{E}_i(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \left( 3 \frac{\vec{P} \cdot \vec{r}}{r^5} - \frac{\vec{P} \cdot \hat{z}}{r^3} \right) = \frac{P}{4\pi\varepsilon_0} \frac{1}{r^3} \left( 3 \frac{\vec{r} \cdot \vec{r}}{r^2} - \hat{z} \right)
\]

**Remarks:**
1. The \( \vec{E} \) field falls off as \( \frac{1}{r^3} \) (the potential falls as \( \frac{1}{r^2} \)) at large distances.
2. For the point dipole, \( q \to \infty \), \( d \to 0 \), \( \vec{P} = q \vec{d} \) finite, the \( \vec{E} \)-field is exactly that obtained in the \( l=1 \) order, i.e., to say that the \( \vec{E} \)-field above is the exact \( \vec{E} \)-field for the point dipole.

In the limit \( q \to \infty \), \( d \to 0 \), \( \vec{P} = q \vec{d} \) finite, one may show that all higher-order terms \( l \geq 2 \) have exactly vanishing multipole moments, as they scale as \( q d^l \to 0 \) for \( l \geq 1 \)

\[
q \to \infty \quad d \to 0 \quad q d \text{ finite}
\]
(3) Even for the point dipole, there is a correction of form, $-\frac{\mathbf{r}}{3\mathbf{E}_0}$, to the $\mathbf{E}$ field, but this only affects the field at the location of the dipole itself.

You may wonder how we missed the delta function term. The answer is that the differentiation in pag 135 is valid except at $r=0$.

(4) The value of the dipole moment of a charge distribution

$$\mathbf{P} = \int \mathbf{r}' \rho(\mathbf{r}') d\mathbf{r}'$$

is generally dependent on the choice of the origin. However when the total charge of the distribution vanishes i.e. there are equal amounts of positive and negative charges, then $\mathbf{P}$ is independent of the origin.

This is more generally true even for a higher-order multipole moment, say of order $l$, provided all of the lower order moments vanishing identically.

\[ l=1 \quad \begin{array}{ccc} +q & \quad \rightarrow \quad & \mathbf{P} \text{ independent of origin, since} \\ & \quad & l=0 \text{ moment, namely total charge} \\ & \quad & \quad \text{is 0!} \\ & \quad & \quad \text{quadrupole moment independent of} \\ & \quad & \quad \text{origin, since } l=0 \text{ and } l=1 \text{ moments} \\ & \quad & \quad \text{namely total charge and total dipole} \\ & \quad & \quad \text{moment vanish identically for the} \\ & \quad & \quad \text{given charge configuration.} \\ \end{array} \]
Prob. 3.33 from the main text

A pure dipole $\mathbf{p}$ is situated in the origin, pointing in the $z$ direction:

a) What is the force on a point charge at $(a, 0, 0)$?

$$\mathbf{F} = q \mathbf{E} = \frac{q \mathbf{p}}{4\pi \varepsilon_0 a^3} (0 - \hat{z}) = -\frac{q \mathbf{p}}{4\pi \varepsilon_0 a^3} \hat{z}$$

b) What is the force on $q$ at $(0, 0, a)$

$$\mathbf{F} = q \mathbf{E} = \frac{q \mathbf{p}}{4\pi \varepsilon_0 a^3} \left( \frac{3a^2}{a^2} \hat{z} \right) = \frac{3q \mathbf{p}}{4\pi \varepsilon_0 a^3} \hat{z}$$

c) How much work does it take to move $q$ from $(a, 0, 0)$ to $(0, 0, a)$?

$$W = q \left[ V(0, 0, a) - V(a, 0, 0) \right] = q \left( \frac{p}{4\pi \varepsilon_0} \frac{1}{a^2} - 0 \right) = \frac{p}{4\pi \varepsilon_0} \frac{1}{a^2}$$

Prob. 3.35 from the main text

A solid sphere, radius $R$, is centered at the origin. The "northern" hemisphere carries a uniform charge density $\rho_0$, and the "southern" hemisphere a uniform charge density $-\rho_0$. Find the approximate field $E(r, \theta)$ for points far from the sphere ($r \gg R$)

$$\mathbf{p} = \int \rho(r) r \, dr$$

$$= \rho_0 \int_{\text{top half}} r \, dr - \rho_0 \int_{\text{bottom half}} r \, dr$$

Clearly, the monopole term is zero.

So we focus on the dipole:

$$\rho_0 \left[ \int_{\text{top half}} r \, dr - \int_{\text{bottom half}} r \, dr \right]$$

$$0 \leq \theta \leq \frac{\pi}{2}, \quad \frac{\pi}{2} \leq \theta \leq \pi$$
with \( \vec{r} = r \sin \theta \cos \phi \hat{x} + r \sin \theta \sin \phi \hat{y} + r \cos \theta \hat{z} \)

(use cartesian basis vectors, as they are constant and can be pulled out of the integrals).

\[
\int r^2 \, dr = \int_0^{\pi/2} \int_0^{2\pi} r^2 \, d\phi \, \sin \theta \, d\theta = \frac{2\pi \sqrt{R^2 - \mu^2}}{4}
\]

Since \( \int_0^\pi \sin \theta \, d\theta = \int_0^\pi \cos \phi \, d\phi \), the first two terms inside the sum in the integrand give vanishing contribution to the integral.

\[
\Rightarrow \int r^2 \, dr = 2\pi \frac{\sqrt{R^2 - \mu^2}}{4} \int_0^\pi \cos \theta \, \sin \theta \, d\theta = \frac{\pi R^4}{4}
\]

By symmetry \( \int r^2 \, dr = -\frac{\pi R^4}{4} \)

\[
\int_0^{\pi/2} \int_0^{2\pi} r^2 \, d\phi \, \sin \theta \, d\theta = \frac{9\pi R^2}{4}
\]

\[
\Rightarrow \vec{P} = \int_0^{\pi/2} \left[ \frac{\pi R^4}{4} \hat{z} - \left( -\frac{\pi R^4}{4} \hat{z} \right) \right] = \frac{\pi R^4}{2} \int_0^{\pi/2} \hat{z} = \frac{3\pi R^4}{4} \hat{z}
\]

For \( r \gg R \) \( E(F) = \frac{1}{4\pi \varepsilon_0} \frac{1}{r^3} \left[ \frac{3qR^3 (\hat{z} \cdot \hat{r}) \hat{r} - 3qR \hat{z}}{4} \right] \)

using the dipole-field formula

\[
\frac{3qR}{16\pi \varepsilon_0} \frac{1}{r^3} \left[ 3 \cos \theta \hat{r} - \hat{z} \right]
\]
Another example

Calculate the monopole, dipole and quadrupole terms of the following charge distribution with:

\[ \lambda = \frac{\varphi}{2\pi R} \]

i) Monopole:

\[
\mathcal{Q} = \int g(r) \, dr = -q + \int_0^{2\pi} R \, d\phi
\]

\[
= -q + \frac{q}{2\pi R} R \int_0^{2\pi} d\phi = -q + q = 0
\]

ii) Dipole

\[
\vec{P} = \int \vec{r} \cdot \rho(r) \, d\mathbf{r} = -q \frac{\mathbf{\hat{y}}}{2} + \int_0^{2\pi} \frac{q}{2\pi R} \left[ 0 \mathbf{\hat{x}} + R \cos \phi \mathbf{\hat{y}} + R \sin \phi \mathbf{\hat{z}} \right] R \, d\phi
\]

\[
= -q \frac{\mathbf{\hat{y}}}{2} + \frac{q}{2\pi R} R^2 \int_0^{2\pi} \left[ \cos \phi \, \mathbf{\hat{y}} + \sin \phi \, \mathbf{\hat{z}} \right] d\phi
\]

\[
= -q \frac{\mathbf{\hat{y}}}{2}
\]

\[
\vec{P} = -q \frac{\mathbf{\hat{y}}}{2}
\]

iii) Quadrupole:

Going back to page 134, we see that

\[
V_2(r) = \frac{1}{4\pi \varepsilon_0} \frac{1}{r^3} \int r^2 \rho(r) \nabla \cdot \mathbf{\hat{n}}(\cos \theta) \, d\mathbf{r}
\]

\[
= \frac{1}{4\pi \varepsilon_0} \frac{1}{r^3} \int r^2 \rho(r) \left( \frac{3 \cos^2 \theta - 1}{2} \right) \, d\mathbf{r}
\]

\[
= \frac{1}{4\pi \varepsilon_0} \frac{1}{r^3} \int \frac{1}{2} \left( 3 \rho(r) \cos^2 \theta - 1 \right) \, d\mathbf{r}
\]

\[
= \frac{1}{4\pi \varepsilon_0} \frac{1}{r^3} \int \frac{1}{2} \left[ \frac{3 \rho(r) \cos^2 \theta - 1}{r^2 \sin^2 \theta} \right] \, d\mathbf{r}
\]

\[
= \frac{1}{4\pi \varepsilon_0} \frac{1}{r^3} \frac{1}{2} \sum_{ij} r_{ij} Q_{ij}
\]

Quadrupole moment

It is a traceless tensor!!
In our case

a) Quadrupole moment due to the point charge

\[ Q_{ij} = \int \rho(r) \left[ \mathbf{r}_i \times \mathbf{r}_j \right] d\mathbf{r} \]

since the position of the charge is \( \mathbf{r}_q = \frac{R}{2} \mathbf{i} \)

then \( Q_{xy} = Q_{xz} = Q_{yz} = Q_{yx} = Q_{yz} = Q_{zx} = 0 \)

\[ Q_{xx} = + \frac{q}{4} \frac{R^2}{4} \]
\[ Q_{yy} = + \frac{q}{4} \frac{R^2}{4} \]
\[ Q_{zz} = - \frac{q}{4} \left[ 3 \left( \frac{R}{2} \right)^2 \left( \frac{R}{2} \right)^2 - \left( \frac{R}{2} \right)^2 \right] = - \frac{2q}{4} \frac{R^2}{4} \]

b) Quadrupole moment of the ring

\[ Q_{ij} = \int d\phi \ A \left[ 3 \ \mathbf{r}_i \times \mathbf{r}_j \right] \]

but \( \mathbf{r}_i = R \cos \phi \ \mathbf{i} + R \sin \phi \ \mathbf{j} \)

\( r^2 = R^2 \)

So \( Q_{xy} = Q_{xz} = Q_{yx} = Q_{yz} = 0 \)

\[ Q_{xx} = \frac{q}{2 \pi R^2} \int_0^{2\pi} d\phi \ ( - R^2 ) = - \frac{q}{2 \pi} \frac{R^2}{2\pi} \]
\[ Q_{yy} = \frac{q}{2 \pi R^2} \int_0^{2\pi} d\phi \ \left[ 3 \cos^2 \phi - 1 \right] R^2 = \frac{q}{2 \pi} \frac{R^2}{2\pi} \]
\[ Q_{zz} = \frac{q}{2 \pi R^2} \int_0^{2\pi} d\phi \ \left[ 3 \sin^2 \phi - 1 \right] R^2 = \frac{q}{2 \pi} \frac{R^2}{2\pi} \]
\[ Q_{yz} = Q_{yz} = \frac{q}{2 \pi R^2} \int_0^{2\pi} d\phi \ 3 \sin \phi \cos \phi \ R^2 = 0 \]

So the total quadrupole moment is

\[ \vec{Q} = \frac{q R^2}{2} \left[ \left( \frac{1}{2} - 2 \right) \mathbf{i}^2 - \left( -1 + 1 \right) \mathbf{j}^2 + \left( \frac{1}{2} + 1 \right) \mathbf{k}^2 \right] = \frac{3q}{4} \frac{R^2}{2} \left[ \mathbf{i}^2 + 2 \mathbf{j}^2 \right] = \frac{3}{4} q R^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]