1. The Cartesian components of any 3-dimensional vector $\vec{A}$ transform under a rotation by the rule

$$A_i' = \sum_{j=1}^{3} R_{ij} A_j, \quad i = 1, 2, 3,$$

where index values 1, 2, and 3 refer to the $x$, $y$, and $z$ components, respectively, of the vector under consideration. The primed coordinates refer to the rotated coordinate system.

(a) Rewrite the above expression using the Einstein summation convention.

(b) Write the above system of equations as a single matrix equation involving two column vectors and the $3 \times 3$ rotation matrix $R$ whose $ij$ element is $R_{ij}$.

(c) Express the scalar product of two vectors, $\vec{A}$ and $\vec{B}$, in matrix form as an appropriate matrix product of a row vector and a column vector.

(d) Show in this way that the scalar product $\vec{A} \cdot \vec{B}$ is invariant under a rotation, i.e., $\vec{A}' \cdot \vec{B}' = \vec{A} \cdot \vec{B}$, provided $R$ is an orthogonal matrix, $R^T R = I$, where $I$ is the $3 \times 3$ identity matrix.

2. Find the gradients of the following functions

(a) $f(x, y, x) = xe^y + z$

(b) $f(x, y, x) = y^4 \ln(x)/z$

3. Assuming that $\vec{r}$ is the separation vector from a fixed point $(x', y', z')$ to the point $(x, y, z)$, and that $r$ is its length. Show that:

(a) $\nabla r^2 = 2\vec{r}$,

(b) $\nabla \frac{1}{r} = -\frac{\vec{r}}{r^2}$,

(c) What is the general formula for $\nabla r^n$?

(d) What is the general formula for $\nabla f(r^n)$?

4. Sketch the vector function

$$\vec{v} = \frac{\vec{R}}{R^3},$$

where $\vec{R} = x\hat{x} + y\hat{y}$, and $R = |\vec{R}|$, and compute its divergence.

5. Find the curl of the following vector function: $\vec{v} = \hat{x}e^{iz}$.

6. The angular-momentum differential operator is defined (up to a complex constant factor) as $\vec{L} = \vec{r} \times \nabla$. Show that $\vec{L} f(r)$, $\vec{L} \cdot \vec{r}$, $\vec{L} \cdot \vec{r} f(r)$, and $\vec{L} \cdot \nabla f(r)$ all vanish, where $f$ is an arbitrary differentiable function of its argument. Beware $r$ is the radial coordinate in spherical coordinates, while $\vec{r}$ is the full position vector.

7. Show that $\nabla^2 (fg) = f \nabla^2 g + 2 \nabla f \cdot \nabla g + g \nabla^2 f$, where $f$ and $g$ are any two differentiable functions.

You may use any approach you like to prove this, whether by working in Cartesian coordinates or by applying in succession the gradient and divergence rules for a product of two functions.