Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy
University of New Mexico

Fall 2004

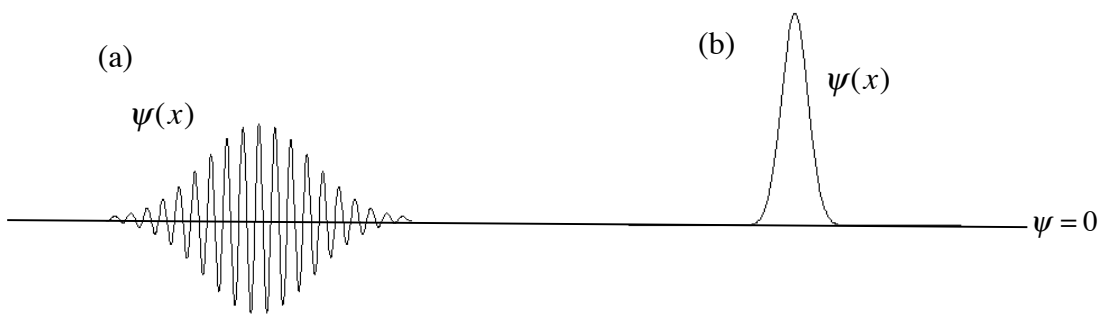
Instructions:

- The exam consists two parts: 5 short answers (6 points each) and your pick of 2 out 3 long answer problems (35 points each).
- Where possible, show all work, partial credit will be given.
- Personal notes on two sides of a 8X11 page are allowed.
- Total time: 3 hours

Good luck!

Short Answers:

S1. Consider a free particle moving in 1D. Shown are two different wave packets in position space whose wave functions are real. Which corresponds to a higher average energy? Explain your answer.



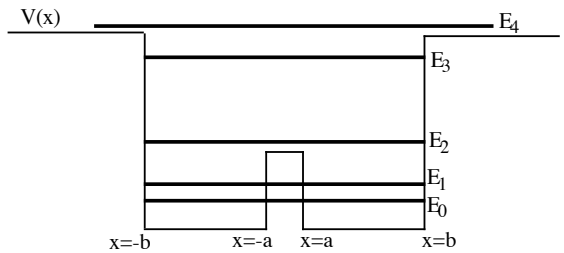
S2. Consider an atom consisting of a muon (heavy electron with mass $m_\mu \approx 200 m_e$) bound to a proton. Ignore the spin of these particles. What are the bound state energy eigenvalues? What are the quantum numbers that are necessary to completely specify an energy eigenstate? Write out the values of these quantum numbers for the first excited state.

S3. Two particles of spins $\vec{s}_1$ and $\vec{s}_2$ interact via a potential $H' = a\vec{s}_1 \cdot \vec{s}_2$.

a) Which of the following quantities are conserved:
$\vec{s}_1, \vec{s}_2, \vec{s}_1{}^2, \vec{s}_2{}^2, \vec{s} = \vec{s}_1 + \vec{s}_2, \vec{s}^{\,2}$?

b) If $s_1 = 5$ and $s_2 = 1$, what are the possible values of the total angular momentum quantum number s?

S4. The energy levels E_n of a symmetric potential well V(x) are denoted below.

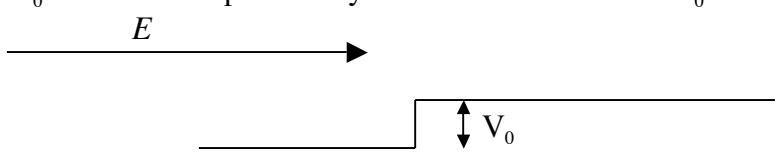


(a) How many bound states are there?

(b) Sketch the wave functions for the first three levels (n=0,1,2). For each, denote the regions where the particle is classically forbidden.

(c) Describe the evolution of the wave function if we start in an equal superposition of eigenstates n=0 and n=1.

S5. A particle with well defined energy E is scattered from dimensional step potential of height V_0. What is the probability of reflection when $E < V_0$ and $E > V_0$.



<u>Long Answers: Pick two out of three problems below</u>

L1. A harmonically bound particle experiences a constant force F.

(a) Argue that the interaction Hamiltonian associated with this force is $\hat{H}_{\text{int}} = -F\hat{x}$.

(b) Assuming the force is weak compared to the harmonic binding, what is the lowest nonvanishing perturbation to the energy of the bound states

(c) Show that this lowest nonvanishing perturbation is

$$\Delta E = -\frac{(Fx_0)^2}{\hbar\omega}, \text{ where } x_0 = \sqrt{\frac{\hbar}{2m\omega}}$$

independent of level. Hint: Recall $\hat{x} = x_0\left(\hat{a} + \hat{a}^\dagger\right)$, where $\hat{a}^\dagger, \hat{a}$ are the creation, annihilation operators.

(d) Solve the Schrödinger equation exactly and find the new energy eigenstates and eigenvalues including the harmonic potential and perturbing force.

(e) Show that your exact energy spectrum agrees with the perturbation result to the appropriate order in the perturbation's small parameter.

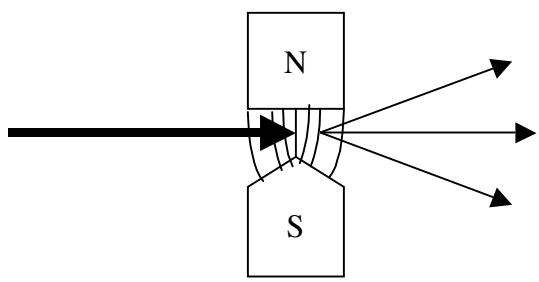
L2. Consider a spin-1 particle. The matrix representations of the components of spin angular momentum are

$$\hat{S}_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \hat{S}_y = \frac{1}{i\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \hat{S}_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

expressed in the ordered basis, $\{|+1\rangle, |0\rangle, |-1\rangle\}$, where the ket is labeled by the eigenvalue of $\hat{S}_z$ in units of $\hbar$.

(a) Show that $|\psi_A\rangle = \frac{1}{2}\left(|+1\rangle - \sqrt{2}|0\rangle + |-1\rangle\right)$, $|\psi_B\rangle = \frac{1}{2}\left(|+1\rangle + \sqrt{2}|0\rangle + |-1\rangle\right)$, and $|\psi_c\rangle = \frac{1}{\sqrt{2}}\left(|+1\rangle - |-1\rangle\right)$ are eigenvectors of $\hat{S}_x$. What are their eigenvalues?

(b) Suppose a beam of particles identically prepared in the state $|\Psi\rangle = \sqrt{\frac{3}{4}}|+1\rangle + \frac{i}{2\sqrt{2}}|0\rangle + \frac{1+i}{4}|-1\rangle$ is put through a Stern-Gerlach apparatus with the gradient B-field along z. Three beams emerge.

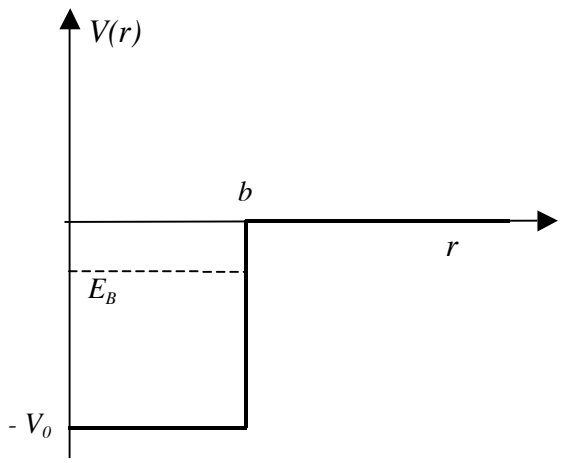


Explain this phenomenon. What are the relative intensities of these beams?

(c) Now suppose the central beam is passed through a second Stern-Gerlach device, oriented with the gradient field along the x-axis.

How many beams will emerge and with what relative intensities?

L3. A deuteron is bound state of proton and neutron ($m_p \sim m_n \sim m \sim 939$ MeV/c^2). It can be modeled as one particle of reduced mass ($\mu = m_p\, m_n/(m_p+m_n)$) in a central square well potential of width b and depth $-V_0$, as shown in the figure. For the deuteron there is only one bound state with a binding energy of $E_B = -2.2$ MeV, which can be assumed to have no orbital angular momentum.



(a) Find the general solution for the reduced radial wave function $u(r) = r\,R(r)$ for the two regions $r<b$ and $r>b$. (You do not need to normalize the wave function)

(b) Show that from the general solution you get a relation between E_B, V_0, b of the form:

$$\tan kb = -\frac{k}{q},$$

where k, q are functions of m, E_B and V_0.

(c) What is the width of the potential (in *fm*) if its depths is $V_0 = 38.5$ MeV? (Use $\hbar c = 197\, fmMeV$ and note that kb must be positive!)

(d) Can there be a second bound state of this potential?

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Fall 2005

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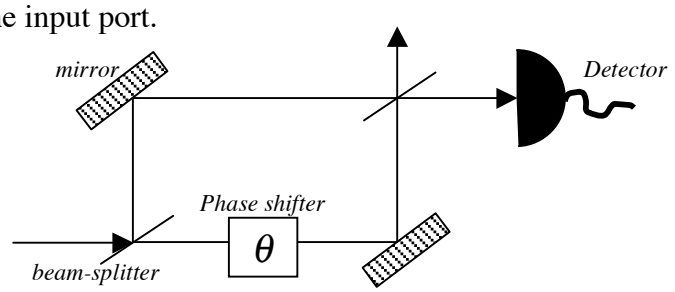
Good luck!

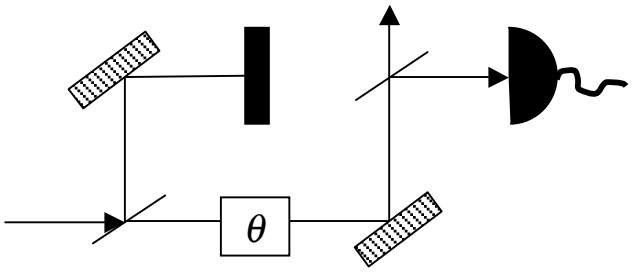
Short answers:

S1. Consider a balanced two-port interferometer sketched below, with 50/50 beam splitters, each arm having a path length *L*. The optical path length of one arm can be made longer through a phase shift θ. A single photon of frequency ω is prepared and injected into the input port.

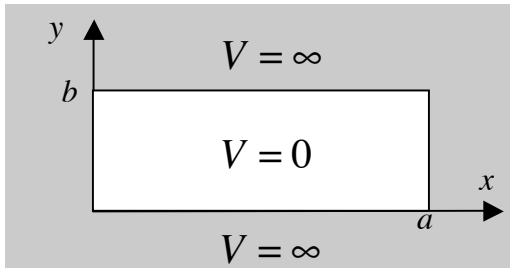


• What is the probability of detecting a photon at the output-port (detector shown) as a function of θ?

• If the other arm of the interferometer is blocked with a completely absorbing barrier (shown below), what is probability of detecting a photon at the denoted output-port vs. θ?



S2. A particle of mass m moves freely in a two-dimensional, infinite height, rectangular potential width a and height b.



What are the energy levels and energy eigenfunctions of the system? Under what conditions are there degeneracies in the spectrum?

S3. A particle is trapped in the ground state harmonic well of frequency ω_0. Its state vector is $|u_0\rangle$. At time t_0 the curvature of the well is changed *suddenly* so that the frequency is doubled. The energy eigenstates of the new well are denoted $\{|\mathrm{v}_n\rangle \mid n = 0,1,...\}$.

- What is the probability of finding the particle in the first excited state of the new well?
- Suppose that the frequency is not changed suddenly, but with some arbitrary time dependence. Describe the condition under which the probability of finding the particle in the new ground state is approximately 1.

S4. A spin 3/2 nucleus is placed in a magnetic field **B** in the z-direction. The nuclear magnetic dipole moment is described by the operator $\vec{\mu} = g\mu_N \vec{\mathbf{I}}$, where μ_N is the nuclear magneton (with g-factor) and $\vec{\mathbf{I}}$ the nuclear spin magnetic moment operator in units of $\hbar$. The Hamiltonian describing the interaction of the spin with the field is $H = -\vec{\mu} \cdot \mathbf{B}$.

- What are the possible eigenvalues and eigenvectors. What is the ground state of the system?
- Roughly describe an experiment you might perform to measure the spectrum of eigenvalues.

S5. The valence shell of He has the configuration $1s^2$. What is the total spin of the ground state? What is the degeneracy of the ground state? Is there any spin-orbit coupling in the ground state?

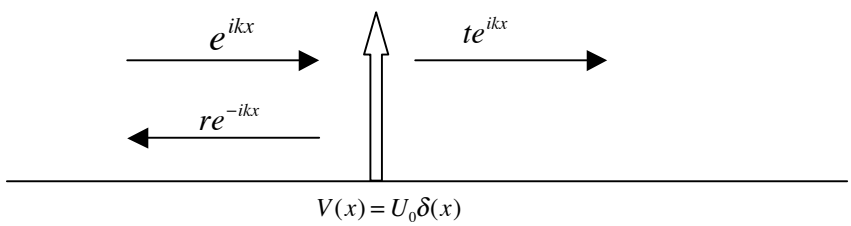
L1. Tunneling and decay

Consider a particle moving in 1D incident on thin potential barrier. We will model this barrier as a repulsive delta-function potential at the origin, $V(x) = U_0\delta(x)$.

(a) What are the units of U_0?
(b) Use the time-independent Schrödinger equation to show that the derivative of the wave function at the origin is discontinuous by,

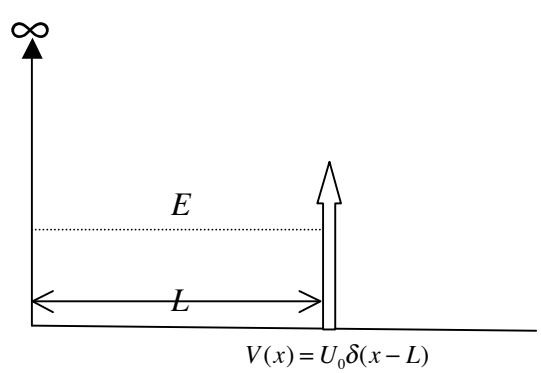
$$\left.\frac{d\psi}{dx}\right|_{0+} - \left.\frac{d\psi}{dx}\right|_{0-} = \frac{2m}{\hbar^2} U_0\, \psi(0).$$

(c) Now consider the scattering of a free particle with energy E. We denote the probability amplitude for transmission t and reflection r.



Show that $r = \dfrac{-i\gamma}{k + i\gamma}$, $t = \dfrac{k}{k + i\gamma}$ where $k = \sqrt{2mE/\hbar^2}$ and $\gamma = \dfrac{mU_0}{\hbar^2}$. Is the sum of the probabilities for reflection and transmission equal to one?

(d) A simple model of α-particle decay is as follows. The confinement of the α-particle in the nucleus is model by a square well of width L. Decay occurs via tunneling through a finite delta-function barrier as above.



For an alpha particle near the ground state, *estimate* the decay rate to free space.

L2. Coherent states of the harmonic oscillator

Consider a harmonic oscillator in one dimension for a particle of mass m and oscillator frequency ω. A special class of wave packets known as "coherent states" can be constructed from a superposition stationary states (defined by $\hat{H}|n\rangle = \hbar\omega(n+1/2)|n\rangle$),

$$|\alpha\rangle = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{(\alpha)^n}{\sqrt{n!}} |n\rangle,$$

where α is a complex number. It has the property that it is an eigenstate of the "annihilation operator" satisfying,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle.$$

(a) Show that the mean position and momentum of the wave packet are

$$\langle \hat{x} \rangle = \sqrt{\frac{2\hbar}{m\omega}} \mathrm{Re}(\alpha), \ \langle \hat{p} \rangle = \sqrt{2\hbar m\omega}\, \mathrm{Im}(\alpha).$$

(b) Show that the uncertainty variances in position and momentum are,

$$\Delta x^2 = \frac{\hbar}{2m\omega}, \ \ \Delta p^2 = \frac{\hbar m\omega}{2}.$$

Is this a minimum uncertainty wave packet? Explain.

(c) Suppose at time t=0 the oscillator is prepared in this wave packet, $|\psi(0)\rangle = |\alpha\rangle$. Show that a later time, the particle is still in a coherent state (up to an overall irrelevant phase) but with $\alpha \to \alpha e^{-i\omega t}$, i.e.

$$|\psi(t)\rangle = |\alpha e^{-i\omega t}\rangle$$

L3. The Stark effect

Consider a hydrogen atom in a uniform static electric field F (we use F here to avoid confusion with the energy E). For a fixed nucleus, and field in z-direction, the perturbation on the electron is described by the Hamiltonian

$$H_1 = eFz = eFr\cos\theta\,,$$

where $z=r\cos\theta$ is the z-coordinate of position of the electron relative to the nucleus (also expressed in spherical coordinates). We ignore spin and any relativistic effects in this problem.

(a) Show that the perturbation has no effect on the energy of the ground state, $1s$, to first order in F. Explain physically.

(b) Show that to second order, the shift on the ground state is

$$\Delta E_{1s}^{(2)} = -2a_0F^2\sum_{n=2}^{\infty}\sum_{l=0}^{n-1}\sum_{m=-l}^{l}\frac{\left|\langle nlm|z|1,0,0\rangle\right|^2}{1-\dfrac{1}{n^2}}\,,$$

where a_0 is the Bohr radius and $|nlm\rangle$ are the unperturbed stationary states of hydrogen with principle quantum number n, angular momentum magnitude l, and projection m.

(c) Given the facts $Y_{0,0}(\theta,\phi)=\sqrt{\dfrac{1}{4\pi}}$ and $Y_{1,0}(\theta,\phi)=\sqrt{\dfrac{3}{4\pi}}\cos\theta$ (spherical harmonics), show that the perturbation simplifies to

$$\Delta E_{1s}^{(2)} = -2a_0F^2\sum_{n=2}^{\infty}\frac{|I_n|^2}{1-\dfrac{1}{n^2}}\,,$$

where

$$I_n = \frac{1}{\sqrt{3}}\int_0^{\infty} dr\; r\, R_{n1}(r)R_{10}(r),$$

with $R_{nl}(r)$ the reduced radial wave function for unperturbed hydrogen.

<u>Preliminary Examination: Quantum Mechanics</u>

Department of Physics and Astronomy

University of New Mexico

Fall 2006

Instructions:

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- Total time: 3 hours

Good luck!

1. A Hydrogen atom is in a state with principle quantum number n = 2.
(4pts) What are the possible values of angular momentum in this state?
(6pts) Consider the state with angular momentum quantum numbers $l = 1$, $m = 1$. Using the Virial Theorem, find the average speed of the electron.

2. The raising and lowering operators for the quantum harmonic oscillator satisfy

$$a^{\dagger}\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$$

$$a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle$$

for energy eigenstates $\left|n\right\rangle$ with energy E_n. Determine the first-order shift in the n = 2 energy level due to the perturbation $\Delta H = V\left(a + a^{\dagger}\right)^2$, where V is a constant.

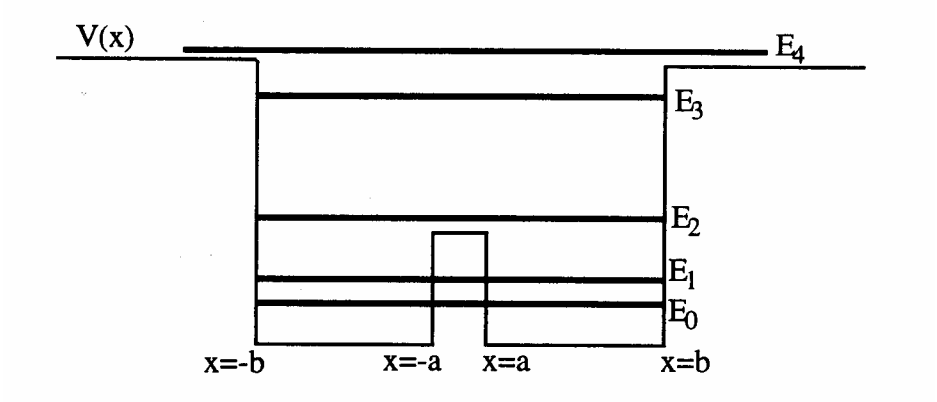
3. For atoms of low to moderate Z, the LS coupling scheme is appropriate. Consider an atom with two optically active electrons with quantum numbers $l_1 = 2$, $s_1 = ½$; $l_2 = 3$, $s_2 = ½$. Find the possible values of l, s, and the resulting values of the total angular momenta j. Be sure to specify which values of j go with each l and s combination.

4. In a photoelectric experiment, monochromatic light of wavelength 400 nm shines on a metal anode with a work function of 2.5 eV. What electric potential between the anode and cathode would be required to stop all of the photoelectrons from reaching the cathode?

If one were now to use monochromatic light of half the wavelength and half the intensity with the same metal anode, what will be the new stopping potential?

5. Imagine a system with just two linearly independent states: $|1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |2\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so that a general state can be written as $|S\rangle = a|1\rangle + b|2\rangle$, *with* $a^2 + b^2 = 1$. Suppose the Hamiltonian for this system has the form: $\mathbf{H} = \begin{pmatrix} h & g \\ g & h \end{pmatrix}$. If the system starts out (at $t = 0$) in state $|1\rangle$, what is its state at time t? Give a physical example of such a system.

6. The energy levels E_n of a symmetric potential well are denoted below:



(1pt) How many bound states are there?

(4pts) Sketch the wavefunctions for the n = 0, 1, 2, and 4 levels. For each, denote the regions where the particle is classically forbidden.

(5pts) Describe the evolution of the wavefunction if we start in an equal superposition of the n = 0 and n = 1 eigenstates.

7. Given a particle with the Hamiltonian $H = \frac{\hat{p}^2}{2m} + V(x)$, find $\frac{d}{dt}\langle \hat{p} \rangle$. Interpret this result.

8. A particle of mass *m* scatters off of a 3-D potential given by:

$$V(r) = \begin{cases} -V_0 & r < R_0 \\ 0 & r \geq R_0 \end{cases}$$

Find the s-wave scattering phase shift for the scattered wave.

9. Consider a 2-D Hilbert space (describing e.g. a spin ½ particle). Let us denote abstractly two different orthonormal basis sets for this space: $\{|+\rangle, |-\rangle\}$ and $\{|\uparrow\rangle, |\downarrow\rangle\}$. The inner products of these states are given by:

$$\langle\uparrow|+\rangle = \langle\uparrow|-\rangle = \frac{1}{\sqrt{2}}, \quad \langle\downarrow|+\rangle = -\langle\downarrow|-\rangle = \frac{1}{\sqrt{2}}.$$

Let $|\psi\rangle = \frac{1}{\sqrt{3}}|+\rangle + \frac{2i}{\sqrt{3}}|-\rangle$ be the state of the particle. Write the state $|\psi\rangle$ as a superposition of the basis vectors $\{|\uparrow\rangle, |\downarrow\rangle\}$. What is the probability of finding the photon in the $|\downarrow\rangle$ state?

10. A spinless particle of mass *m* resides in two dimensions and experiences the potential $V(x,y)=\frac{1}{2}k\left(x^2+y^2\right)$, with k a positive constant. What are the conserved quantities for this potential? Give the energies and degeneracies of the first two energy levels.

Preliminary Examination: Quantum Mechanics
Department of Physics and Astronomy
University of New Mexico
Spring 2007

Instructions:
- The exam consists of 10 problems, 10 points each.
- Partial credit will be given if merited.
- Personal notes on two sides of an 8 × 11 page are allowed.
- Total time is 3 hours.

Problem 1: A particle in a potential well is in a superposition of bound states, described by the wave function

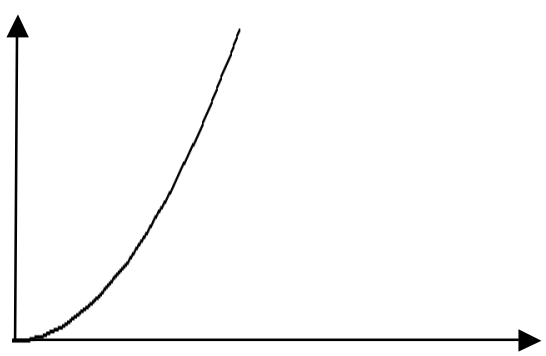
$$\psi(x) = \sqrt{\frac{2}{3}}u_0(x) + \frac{1+i}{\sqrt{12}}u_2(x) + \frac{e^{-i\pi/4}}{\sqrt{6}}u_3(x),$$

where $u_n(x)$ are the orthonormal energy eigenfunctions satisfying, $Hu_n(x) = E_n u_n(x)$, (n=0 ground state, n>0, n^{th} excited state).

(i) Show that the wave function is normalized.
(ii) If energy is measured, what values are possible and with what probability?
(iii) What is the expectation value of energy in terms of the eigenvalues, E_n?

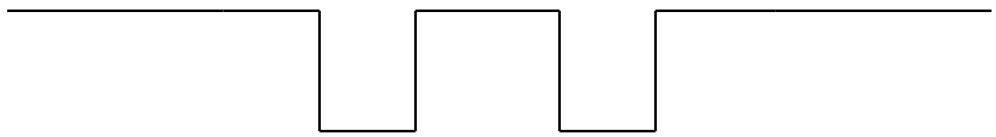
Problem 2: A particle of mass m is in a one-dimensional well described by,

$$V(x) = \begin{cases} \frac{1}{2}kx^2, & x > 0 \\ \infty, & x < 0 \end{cases}$$

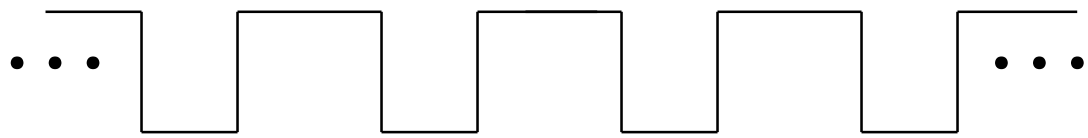


(i) Sketch the ground state and first excited state wave function in the well.
(ii) What are the energy eigenvalues?

Problem 3: Consider a double-well consisting of two-finite square wells, separated by a thick barrier, sketched below.

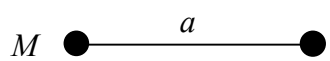


(i) Assuming each well supports 2 bound states in isolation (i.e., with an infinitely thick barrier), sketch the energy levels and eigenfunctions for the double well.
(ii) If we now have a periodic train of wells, describe the nature of the energy spectrum.



Problem 4: Show that the ground state of a one-dimensional harmonic oscillator is a minimum uncertainty wave packet, i.e., $\Delta x \Delta p = \hbar/2$.
<u>Hint:</u> Recall that the position and momentum operators can be expressed in terms of creation and annihilation operators, $\hat{x} = \sqrt{\hbar/(2m\omega)}\left(\hat{a} + \hat{a}^{\dagger}\right)$, $\hat{p} = -i\sqrt{\hbar m\omega/2}\left(\hat{a} - \hat{a}^{\dagger}\right)$.

Problem 5: To coarsest approximation, a homonuclear diatomic molecule is described by a rigid rotator, with equal mass nuclei M, attached to a rigid rod of length a.



(i) Taking the center of mass to be fixed, show that the energy eigenvalues are, $E_l = \hbar^2 l(l+1)/Ma^2$; l=0,1,2… (Hint: write the appropriate Hamiltonian first based on your knowledge of the classical problem).
(ii) What are eigenfunctions and degeneracy of the lth level?

Problem 6: A "hydrogenic atom" consists of a single electron orbiting a nucleus with Z protons (Z=1 is hydrogen, Z=2 is singly ionized helium, Z=3 is doubly ionized lithium, etc.).

(i) What are the energy levels as a function of the principle quantum number n, for a given Z?
(ii) What is the degeneracy of the level n (include electron spin, but ignore nuclear spin)?
(iii) Given that the Lyman-alpha line (transition from n=2 to n=1) in hydrogen lies in the far ultraviolet ($\lambda = 121.6$ nm), what part of the spectrum does the Lyman-alpha lie in for Z=2 and Z=3?

Problem 7: Consider a hydrogen atom in a uniform electric field $\mathcal{E}$ in the z-direction. The perturbation Hamiltonian is $\hat{H}_1 = -e\mathcal{E}\hat{z}$, where $\hat{z}$ is the z-coordinate operator of the electron relative to the nucleus.
(i) Show that the shift in the ground state vanishes to first order in the perturbation
(ii) Write an expression for the second order shift of the ground state.

Problem 8: The dynamics of a spin-1/2 electron placed in a magnetic field B in the z-direction is governed by the Hamiltonian $\hat{H} = \mu_B B \hat{\sigma}_z$, where μ_B is the Bohr magneton and $\hat{\sigma}_z$ is the z-component Pauli spin-operator. A time t=0 the spin is prepared as spin-up along x, which can be expressed as a superposition of spin-up and down-up along z,

$$|\psi(t=0)\rangle = |\uparrow_x\rangle = \left(|\uparrow_z\rangle + |\downarrow_z\rangle\right)/\sqrt{2}.$$

What is the probability of finding the electron in $|\uparrow_x\rangle$ at a later time t?

Problem 9: Consider two spin-1/2 particles, labeled A and B.
The familiar complete set of commuting operators

$$C_1 = \left\{\hat{\vec{\mathbf{S}}}^2 = (\hat{\vec{\mathbf{s}}}_A + \hat{\vec{\mathbf{s}}}_B),\ \hat{S}_Z = \hat{s}_{Az} + \hat{s}_{Az},\ \hat{\vec{\mathbf{s}}}_A^2,\ \hat{\vec{\mathbf{s}}}_B^2\right\},$$

leads to the "coupled basis" of triplets and singlet. An alternative set that leads to the so-called "Bell basis" is

$$C_2 = \left\{\hat{\vec{\mathbf{s}}}_A^2,\ \hat{\vec{\mathbf{s}}}_B^2,\ \hat{\sigma}_{Ax}\hat{\sigma}_{Bx},\ \hat{\sigma}_{Az}\hat{\sigma}_{Bz}\right\}.$$

Show that these two sets are not compatible. Note: $\vec{\mathbf{s}} = \frac{\hbar}{2}\vec{\sigma}$.

Problem 10: Show that the Bell-basis, defined by the four kets,

$$|\Phi^{(\pm)}\rangle = \left(|\uparrow_z^A, \uparrow_z^B\rangle \pm |\downarrow_z^A, \downarrow_z^B\rangle\right)/\sqrt{2}, \text{ and } |\Psi^{(\pm)}\rangle = \left(|\uparrow_z^A, \downarrow_z^B\rangle \pm |\downarrow_z^A, \uparrow_z^B\rangle\right)/\sqrt{2},$$

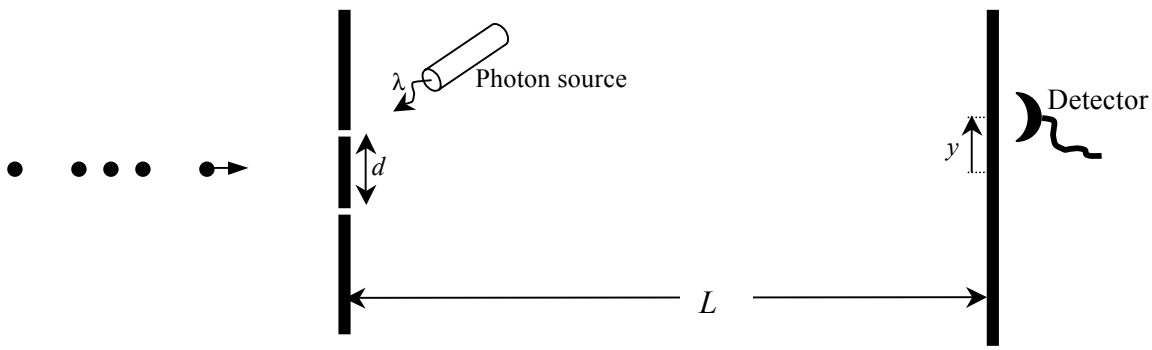
are simultaneous eigenvectors of the set C_2 defined in problem 9. What are the eigenvalues of all four operators?

Hint: Recall, that for a given spin, $\hat{\sigma}_z = |\uparrow_z\rangle\langle\uparrow_z| - |\downarrow_z\rangle\langle\downarrow_z|$, $\hat{\sigma}_x = |\uparrow_z\rangle\langle\downarrow_z| + |\downarrow_z\rangle\langle\uparrow_z|$.

Preliminary Examination: Quantum Mechanics
Department of Physics and Astronomy
University of New Mexico
Fall 2008

Instructions:
• The exam consists of 10 problems, 10 points each.
• Partial credit will be given if merited.
• Personal notes on two sides of an 8 × 11 page are allowed.
• Total time is 3 hours.

Problem 1: A highly collimated, nearly monoenergetic beam of electrons, energy E, is incident on mask with two very thin long slits, separated by distance d. The electrons are detected downstream art a distance L, very far away, $L>>d$.

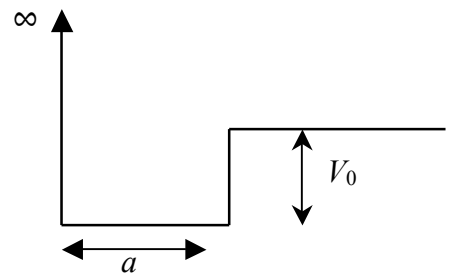


(i) What is the probability density (unnormalized) of detecting an electron as a function of the position of the detector y, and any other relevant parameters?
(ii) We know that measuring the system can affect the wave function, and the probabilities of outcomes of subsequent measurements. If a photon is scattered off the electrons just after they pass the slits, estimate the minimum wavelength of that light such that the interference pattern disappears.

Problem 2: A very cold gas of identical bosons will condense into a Bose-Einstein-Condensate (BEC) when the average spacing between the particles is equal to the deBroglie wavelength.

(i) Given a gas at density n in three dimensions, find an approximate formula for the critical temperature at which the gas condenses to BEC in terms of n, the mass of the particles m, Planck's constant, and Boltzmann's constant.
(ii) A composite particle like an atom can be boson and condensed to BEC. The isotope of lithium (Z=3) ^{7}Li is a boson while the isotope ^{6}Li is fermion. Explain how this can be.

Problem 3: Consider a particle of mass m moving in a one dimensional well with an infinitely high wall at the origin, and a wall of potential height V_0 at $x=a$.



What is the minimum height V_0 such that the well will support at least one bound state.

Problem 4: Consider a simple harmonic oscillator for a particle of mass m and frequency ω. Find the uncertainty product $\Delta x \Delta p$ for the nth energy eigenstate (with n=0 the ground state). For what n is this the minimum possible value?

Problem 5: Consider a diatomic molecule of atoms with masses m_1 and m_2. We can model the binding potential between the atoms by a spherically symmetric "Lennard-Jones" potential $V(r) = C_{12}/r^{12} - C_6/r^6$, where C_{12}, C_6 are constants, and r is the relative distance between the atoms.

(i) Ignoring any internal structure of the atoms, write a Hamiltonian that describes their relative motion in terms of the Lennard-Jones potential, the relative-coordinate angular momentum, and any other relevant parameters.
(ii) Close to the ground state, the potential looks like a harmonic well. In that regime, and for zero angular momentum, what are the approximate energy levels of the molecule? You may express your answer in terms of the Lennard-Jones potential and its derivatives.

Problem 6: Consider a spin-1 particle, spin operator **S**. In the basis of eigenstates of S^2 and S_z, ordered as $\{|S=1, m_s=1\rangle, |S=1, m_s=0\rangle, |S=1, m_s=-1\rangle\}$, the matrix representations of the three components of spin-angular momentum are

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_y = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad S_z = \hbar\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(i) Show that these matrices satisfy the expected commutation relations for angular momentum (you need only pick one commutator to show).
(ii) What are the eigenvalues of S_x?

Problem 7: The hyperfine interaction between electron and nuclear spin is described by a Hamiltonian of the form $H = A\mathbf{I}\cdot\mathbf{S}$, where **I** is the nuclear spin angular momentum, and **S** the electron spin operator.

(i) Ignoring any other effects, what are the good quantum numbers that specify the eigenstates of this Hamiltonian.
(ii) What are the eigenvalues of H in terms of these quantum numbers, and what are the degeneracies of the energy levels?

Problem 8: Two non-interacting spin ½ identical particles are placed in a one-dimensional harmonic well.
(i) Write the two-particle ground state including motional and spin degrees of freedom.
(ii) A magnetic field **B** is applied. What is the energy shift to first order in **B** due to the interaction with the magnetic moment of the spins?

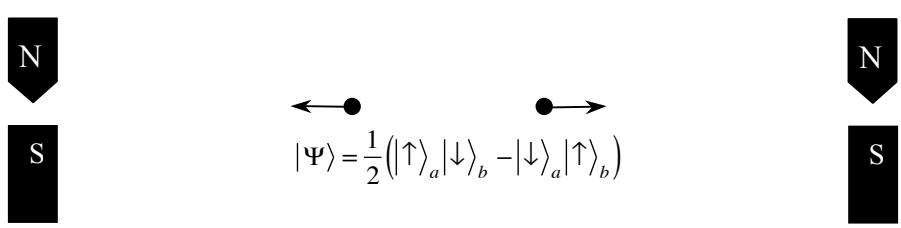
Problem 9: A particle of mass m is placed in a three dimensional, isotropic harmonic well, described by potential, $V(x,y,z) = \frac{1}{2}m\omega^2\left(x^2 + y^2 + z^2\right)$.

(i) What is the energy and degeneracy of the first excited state?

A perturbation potential of the form $V_1(x,y) = Axy$ is now added.

(ii) Calculate the shifts of the sublevels of the first excited state to lowest nonvanishing order in perturbation theory?

Problem 10: Two electrons a and b are in a spin-singlet state $|\Psi\rangle = \frac{1}{2}\left(|\uparrow\rangle_a|\downarrow\rangle_b - |\downarrow\rangle_a|\uparrow\rangle_b\right)$, where $|\uparrow\rangle, |\downarrow\rangle$ are spin-up and down along the z-axis. The two electrons fly apart and are analyzed in separated Stern-Gerlach apparatuses.



(i) If electron-a is measured along the **z-axis** and found to be spin-up, what is the probability for electron-b to be found spin-up **along z**?
(ii) If electron-a is measured along the **x-axis** and found to be spin-up, what is the probability for electron-b to be found spin-up **along x**?
(iii) If electron-a is measured along the **z-axis** and found to be spin-up, what is the probability for electron-b to be found spin-up **along x**?

Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy
University of New Mexico

Fall 2009

Instructions:

- *the exam consists of 10 problems, 10 points each;*
- *partial credit will be given if merited;*
- *personal notes on two sides of* 8×11 *page are allowed;*
- *total time is 3 hours.*

Table of Constants and Conversion factors

Quantity	Symbol	$\approx$ value
speed of light	c	3.00×10^8 m/s
Planck	h	6.63×10^{-34} J·s
electron charge	e	1.60×10^{-19} C
e mass	m_e	0.511 Mev$/c^2 = 9.11 \times 10^{-31}$ kg
p mass	m_p	938 Mev$/c^2 = 1.67 \times 10^{-27}$ kg
fine structure	α	1/137
Boltzmann	k_B	1.38×10^{-23} J/K
Bohr radius	$a_0 = \hbar/(m_e c\alpha)$	0.53×10^{-10} m
conversion constant	$\hbar c$	200 eV·nm
conversion constant	k_BT@ 300K	1/40 eV
conversion constant	1eV	1.60×10^{-19} J

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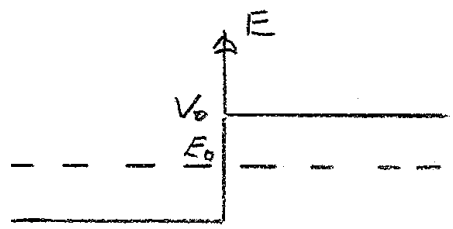
1. At what speed is the deBroglie wavelength of an alpha particle equal to that of a 10 KeV photon.
2. Argue using the deBroglie wavelength whether or not QM is needed to describe the physical system:
   a) atomic electron (note that for the ground state of hydrogen $\langle v \rangle = c\alpha$)
   b) hydrogen gas with number density $10^{21}/\text{cm}^3$ at T=300K
3. Three essential features of the non-relativistic Schroedinger equation are that it is linear, it depends on the first time derivative and it explicitly contains the complex number i in the time derivative term. State the physical consequences of each of these features.
4. A particle is confined to a 1D box $0 < x < a$ and at $t = 0$ has the wave function

$$\psi = Ax(a - x).$$

   (a) Determine the normalization constant A.
   (b) Find the expectation value of the energy for this state.
   (c) How does your result in (b) compare qualitatively ($>, =, <$) with the ground state energy and explain your reasoning.

5. A particle moving in 1D with kinetic energy E_0 is incident on an infinitely wide potential barrier with $V_0 > E_0$. Find the penetration depth of the particle into the barrier.



6. A photon is in the state

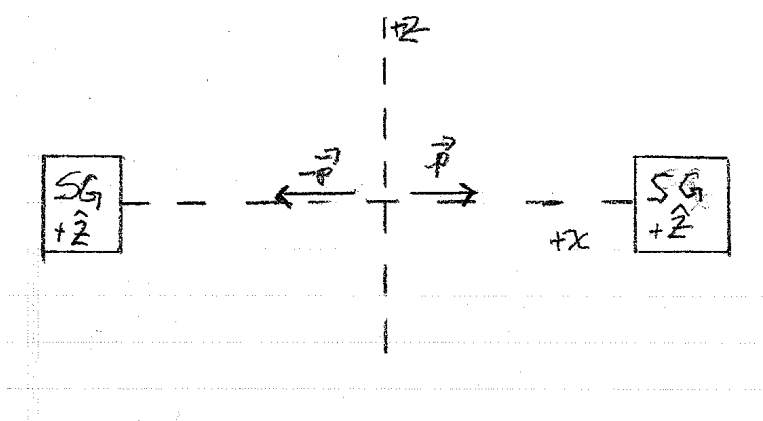
$$|\psi\rangle = \frac{1}{5}(3|\underline{x}\rangle + i4|\underline{y}\rangle$$

where $|\underline{x}\rangle$, $|\underline{y}\rangle$ refer to photon states polarized along the x and y axes respectively.
(a) What is the probability of the photon being right circularly polarized?
(b) Suppose the photon passes through a plate that introduces a relative phase shift of π between the states $|\underline{x}\rangle$, $|\underline{y}\rangle$. What is the probability of the photon being right-circularly polarized now?

7. Consider two electrons with equal and opposite momenta directed along the x-axis and produced in the entangled spin state

$$\frac{1}{\sqrt{2}}(|\Uparrow_z, \Downarrow_z\rangle - |\Downarrow_z, \Uparrow_z\rangle).$$

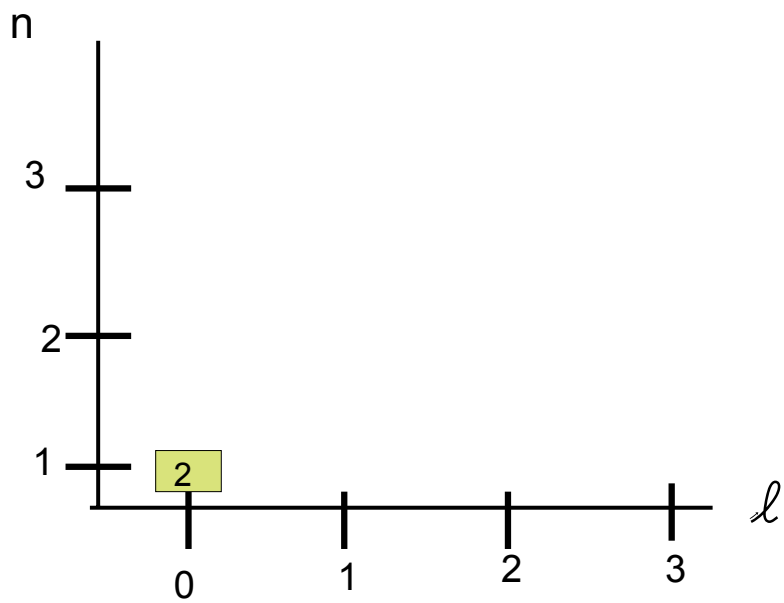
The electron with momentum $\vec{p}$ is the first in the product ket, and the $-\vec{p}$ is the second. This is sketched in the figure below, where the boxes labeled "SG $+\hat{z}$" represent Stern-Gerlach devices that deflect electrons in the $|\Uparrow_z\rangle$ state in the $+\hat{z}$ direction and electrons in the $|\Downarrow_z\rangle$ state in the $-\hat{z}$ direction thereby measuring the electron spin components along the z direction.



a) What are the possible outcomes for the two SG measurements? What are the probabilities for the outcomes given this entangled state.
b) What is the total spin of this state? (Think of the symmetry of this state and how you would write down the other possible two-electron product states.)
c) The experiment is repeated with SG devices oriented in the y direction thereby measuring the spin components along the y axis. What is the probability for each of the possible outcomes now? (No calculations needed if you use the symmetry property of this state.)

8. Calculate the first order correction to the energy of a simple 1D harmonic oscillator with displacement in the x direction due to the perturbation bx^4.

9. a) For the $1/r$ potential (simple hydrogen atom) the states are labeled by quantum numbers n,ℓ,m_ℓ,s,m_s. The states with magnetic quantum number m_ℓ but the same ℓ are degenerate. Why? Same question for the spin quantum numbers s,m_s.
   b) A typical energy level (not to scale) diagram is shown below, where each box represents states of the same n,ℓ and is labeled by the degeneracy including spin (number of states in the box). Complete the diagram up to n=3.



10. Add the spin-orbit coupling to the simple hydrogen atom. For the $n = 2$ level, label **all** the states according to the "good" quantum numbers. Show that the total number of states sums to $2n^2 = 8$ (2 for spin).

Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy
University of New Mexico

Fall 2010

Instructions:

- *the exam consists of 10 problems, 10 points each;*
- *partial credit will be given if merited;*
- *personal notes on two sides of* 8×11 *page are allowed;*
- *total time is 3 hours.*

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1. A polished silicon surface is seen by a free neutron as an (infinitely sharp) impenetrable barrier. Under the influence of gravity, unlike a classical particle that rests on a hard horizontal surface, the neutron will hover above the surface.
   (a) Sketch the wave function of the ground state of such a neutron.
   (b) Derive an approximate expression for the expectation value of the height of the neutron above the silicon using the uncertainty principle (in terms of the acceleration of gravity g, the mass of the neutron m_n, and other constants).
2. Consider a particle of mass m in a one-dimensional potential well $V(x) = -V_0$ for $|x| < a$ and equal to zero otherwise. Show that at least one bound state will always exist.
3. Consider a Hamiltonian
   $$\hat{H} = \hat{H}_1 + \hat{H}_2.$$
   (a) When can the energy eigenvalue of the total Hamiltonian $\hat{H}$ be written as the sum of eigen-energies E_i for the $\hat{H}_i$ ($i = 1, 2$),
   $$E = E_1 + E_2?$$
   (b) For a diatomic molecule, what is the rigid rotor approximation and show explicitly that E can be written as the sum of vibrational and rotational energies.
4. Find the reflection coefficient for 1 dimensional scattering of a particle of mass m off of a delta function potential $V = -a\delta(x)$.
5. Consider a complex potential $V = V_0 - i\Gamma$ where V_0 and Γ are real constants with dimensions of energy. Use the one dimensional Schrödinger equation to obtain an equation for the probability density and show that the total probability $P = \int \psi^\star \psi dx$ is not conserved; Find dP/dt.
6. An electron is in the spin-up state with respect to a uniform magnetic field. The field direction is suddenly rotated at t=0 by 60°. What is the probability to measure the electron in the spin-up state with respect to the new field direction immediately after the field is rotated?

7. The spin of an electron interacts with a uniform, static magnetic field with

$$\hat{H}_0 = \omega_0 \hat{S}_z.$$

At $t = 0$ the spin state is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)),$$

with the notation $|\uparrow\rangle = |\frac{1}{2}, +\frac{1}{2}\rangle, |\downarrow\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$.

Find $\langle \hat{S}_x \rangle$ as a function of time t.

8. For the two-dimensional harmonic oscillator, the unperturbed Hamiltonian is given by

$$\hat{H}_0 = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{1}{2} m\omega^2 \left(\hat{x}^2 + \hat{y}^2\right).$$

Use perturbation theory to determine the first-order energy shifts to the ground state and the (degenerate) first-excited states due to the perturbation

$$\hat{H}_1 = 2b\hat{x}\hat{y}.$$

9. Consider a particle of mass m in an **<u>infinite</u>** spherical well of radius a. Determine the energy eigenstates and eigenvalues for $\ell = 0$. The energy eigenstates should be written in terms of a normalization constant (A) that you leave undetermined.

10. Consider the two-electron states of Helium in perturbation theory, where we take the e-e Coulomb interaction to be the perturbation.
a) What is the unperturbed energy and degeneracy of the multiplet of first excited states?

Explicitly construct these states in terms of single electron hydrogen atom (spatial) energy eigenstates $|n_i, \ell_i, m_i\rangle_i$ and spinors $|\pm\rangle_i \equiv |\frac{1}{2}, \pm\frac{1}{2}\rangle_i$ (where $i = 1, 2$ labels the electron) for states with **<u>total</u>** orbital angular momentum quantum number ℓ and **<u>total</u>** spin quantum number s for the cases:
b) $\ell = 0$, $s = 1$; and
c) $\ell = 1$, $s = 0$.

Department of Physics and Astronomy, University of New Mexico

Quantum Mechanics Preliminary Examination

Fall 2011

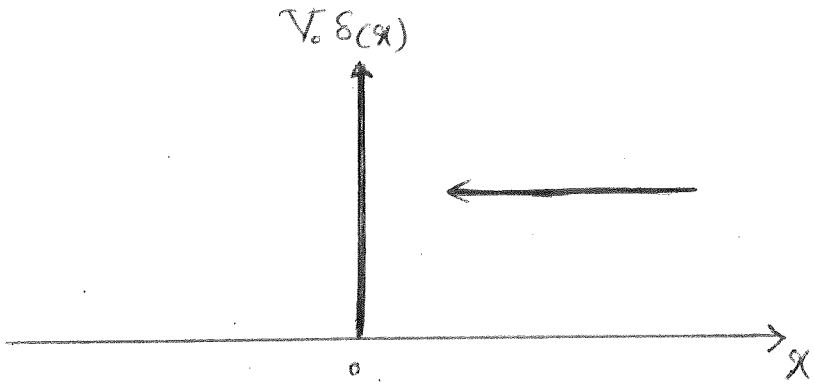
Instructions:

- The exam consists of 10 short-answer problems (10 points each).
- Partial credit will be given if merited.
- Personal notes on two sides of an $8\frac{1}{2}'' \times 11''$ page are allowed.
- Total time: 3 hours.

1– An electron microscope uses a beam of electrons to produce a magnified image of the specimen. Find the minimum kinetic energy of electrons so that the microscope has enough resolution to observe micron size objects.

2– A free particle of mass m is arranged in a Gaussian wavepacket with initial values $\langle X\rangle_0 = 0$, $\Delta X_0 = a$, and $\langle P\rangle_0 = p_0$ at $t = 0$. What is ΔP_0? Find $\langle X\rangle(t)$, $\langle P\rangle(t)$, and $\Delta P(t)$. Describe in words the behavior of $\Delta X(t)$.

3– A particle with mass m and energy E moves in one dimension and approaches a localized potential barrier $V = V_0\delta(x)$ from the right. Find the particle wavefunction (up to an overall constant) and the transmission probability.



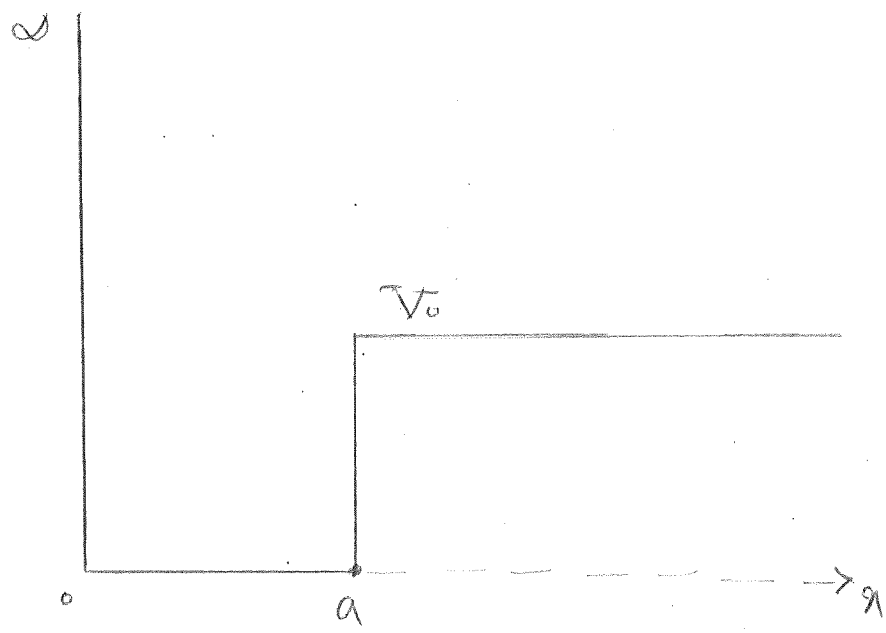
4– A simple harmonic oscillator with mass m and frequency ω is in its ground state. The frequency is suddenly changed to ω'. Find the average energy of the oscillator right after the change in frequency.

5– The Hamiltonian for a system consisting of two nonidentical spin-1/2 particles is given by

$$H = a\vec{S}_1 \cdot \vec{S}_2 + b(S_{1z} + S_{2z}) \,.$$

The first term represents spin-spin interaction between the particles, and the second term comes from the interaction of particles with an external magnetic field in the z direction. Considering only the spin degrees of freedom, find the energy eigenstates and their corresponding eigenvalues for this system.

6– A particle of mass m moves in an asymmetric one-dimensional potential well. The wall at the origin is infinitely high, and the potential height of the wall at $x = a$ is V_0. Does this system always have a ground state? Explain.



7– Consider a simple harmonic oscillator in one dimension with mass m and frequency ω. A perturbation of the form λx^4 is added to the potential. Find corrections to the energy and wavefunction of the ground state to first order in perturbation theory.

8– Consider two spin-1/2 particles freely moving in three dimensions. In the center-of-mass frame, the wavefunction of this system can be written as the product of an orbital angular momentum part and a spin part. From the addition of angular momentum we know that one can obtain total angular momentum $J = 1$ for $L = 0$ & $S = 1$, $L = 1$ & $S = 0$, $L = 1$ & $S = 1$. Which of these are acceptable if the particles are identical? Explain.

9– An electron initially in the eigenstate of $\hat{S}_y$ with eigenvalue $\hbar/2$ is placed in a uniform magnetic field $B = B_0\hat{z}$. What is the probability that the electron is found in the eigenstate of $\hat{S}_y$ with eigenvalue $-\hbar/2$ at a later time t?

10– Consider the $n = 3$ energy level of the Hydrogen atom. What is the total number of states at this level (include electron spin, but ignore spin of the proton)? The presence of a magnetic field $\vec{B} = B_0\hat{z}$ results in an additional term

$$H_B = \frac{eB}{2mc}(L_z + 2S_z)\,,$$

in the Hamiltonian. In the weak field limit, how do the energy of the states with $n = 3$ and their degeneracy affected?

Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy
University of New Mexico

Fall 2012

Instructions:

- *the exam consists of 10 problems, 10 points each;*
- *partial credit will be given if merited;*
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Table of Constants and Conversion factors

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conversion constant	k_BT@ 300K	1/40 eV
conversion constant	1eV	1.60×10^{-19} J

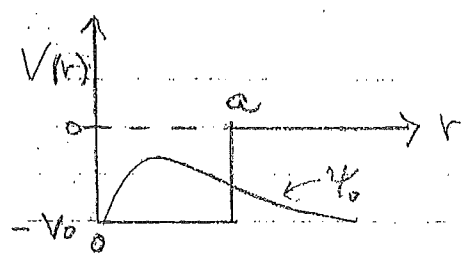
The Pauli spin matricies:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Angular momentum ($j = 0, \frac{1}{2}, 1, ...$):

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z$$

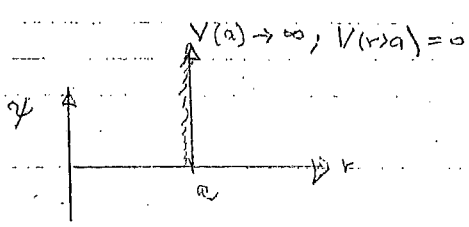
1. Consider a time independent but complex potential $V(x) = V_0(x) - i\Gamma$ where Γ is independent of both space and time. Use the Schrödinger equation in 1D to show that the probability density is not conserved and find the time dependence of the wave function normalization (total probability).
2. For a 3D spherical well of depth V_0 and radius a.
   (a) A bound state wave function is sketched on the figure below. On the same graph, sketch the ground state wave function in the limiting case wherein the particle is bound with infinitesimal energy.
   (b) Find the minimum depth V_0 for the existence of a bound state. The sketch you made in (a) suggests a simple way to find the solution. If you must, you can always resort to solving the transcendental equation.



3. A particle with low momentum $\hbar k$ is scattered by a hard sphere of radius a.
   (a) The radial wave function for the $\ell = 0$ partial wave is of the form $u(r) = rR(r) = \frac{1}{k}\sin(kr + \delta_0)$ where δ_0 is the zeroth partial wave phase shift. Sketch the wave function on the figure below and determine δ_0.
   (b) What is the $k \to 0$ total cross section limit? Recall the total cross section is

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell}(2\ell+1)\sin^2\delta_\ell$$

   where δ_ℓ is the ℓth partial wave phase shift.



4. Consider two electrons produced in an entangled spin state with total spin $S = 0$ (spin singlet state). What is the probability to measure one electron with spin-up along the direction $\hat{a}$ and the other electron with spin-up along the direction $\hat{b}$, where these directions are arbitrary?
5. Consider an electron bound to a metal with work function W. A constant electric field exists outside the metal, giving a potential $V(x) = W - e\mathcal{E}x$ for $x > 0$ where e is the absolute value of the electron charge and $\mathcal{E}$ is the magnitude of the electric field, and the direction $\hat{x}$ is taken normal to the metal surface. Find the tunneling transmission coefficient (T) for an electron at the Fermi energy (i.e. bound by energy W). You should get that $\log T \propto W^{3/2}/\mathcal{E}$. Recall that the transmission coefficient is of the form

$$T = \exp\left(\frac{-2}{\hbar}\int |p(x)|dx\right),$$

where $p(x)$ is the momentum.

6. A particle of mass m in a one-dimensional harmonic oscillator with angular frequency ω is prepared in a state where an energy measurement yields $\hbar\omega/2$ or $3\hbar\omega/2$ each with probability 1/2. If we also require that $\langle p\rangle = \sqrt{\hbar m\omega/2}$ at $t = 0$ the state is completely specified. Recall

$$\hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right).$$

(a) What is the complete normalized state as a function of time?
(b) What is $\langle p\rangle$ as a function of time?

7. The spinor (spin-$\frac{1}{2}$) rotation operator is given by

$$R(\alpha\hat{a}) = I\cos\left(\frac{\alpha}{2}\right) - i\hat{a}\cdot\vec{\sigma}\sin\left(\frac{\alpha}{2}\right)$$

about the direction $\hat{a}$ by angle α, and where σ_i, $i = x, y, z$ are the Pauli spin matricies and I is the 2×2 unit matrix. Use this to construct the state $|+n\rangle$ corresponding to the particle having spin $+\frac{1}{2}$ along the direction given by the Euclidean vector $\hat{n} = \cos(\phi)\sin(\theta)\hat{x} + \sin(\phi)\sin(\theta)\hat{y} + \cos(\theta)\hat{z}$.

8. For a hydrogen atom in an electric field of strength $\mathcal{E}$ in the z direction has a perturbation $\hat{H}' = e\mathcal{E}z$. Show that there is no correction to the ground state to first order in perturbation theory.
9. A hydrogen atom in the presence of a magnetic field will have an additional interaction (Zeeman effect)

$$\hat{H}_{Zeeman} = \frac{\mu_B}{\hbar}\vec{B}\cdot\left(\hat{\vec{L}} + 2\hat{\vec{S}}\right)$$

where $\mu_B = 5.8 \times 10^{-5}$eV/T is the Bohr magneton.
(a) What condition on B constitutes the weak field regime? What should B be compared to?
(b) Find the corrections to the ${}^2P_{\frac{1}{2}}$ doublet ($j = 1, \ell = 1, s = 1/2,$) in the weak field approximation. Recall the explicit form of the states is:

$$|\frac{1}{2},\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|1,1\rangle|\frac{1}{2},-\frac{1}{2}\rangle - \sqrt{\frac{1}{3}}|1,0\rangle|\frac{1}{2},\frac{1}{2}\rangle$$

$$|\frac{1}{2},-\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|1,0\rangle|\frac{1}{2},-\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}|1,-1\rangle|\frac{1}{2},\frac{1}{2}\rangle$$

where the notation refers to $|j, m_j\rangle$ (the states of definite j) on the left hand side and $|1, m_\ell\rangle|\frac{1}{2}, m_s\rangle$ are the product of orbital and spin wave functions on the right hand side.
10. Use the variational method to estimate the ground state energy of the hydrogen atom ($V(r) = -e^2/r$). As a trial wave function take

$$\psi(\vec{r}) = \frac{A}{b^{3/2}}\left(1 - \frac{r}{b}\right), \text{ for } r < b,$$
$$\psi = 0 \text{ for } r > b$$

where the variational parameter b is a length and the dimensionless constant $A = \sqrt{30/4\pi}$. Notice that $\psi^*\psi d^3r$ is dimensionless as it should be. Express the variational parameter b in terms of the Bohr radius $a_0 = \hbar^2/me^2$ and the estimated energy in terms of $e^2/a_0 = 2(13.6eV)$ Recall the Laplacian in spherical coordinates is:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy
University of New Mexico

Fall 2013

Instructions:

•The exam consists of 10 short-answer problems (10 points each).
•Where possible, show all work; partial credit will be given if merited.
•Personal notes on two sides of an 8×11 page are allowed.
•Total time: 3 hours.

Useful Formulae and Constants:

$$\int_{-\infty}^{\infty} dx \exp(-ax^2)x^{2n} = \frac{(2n-1)!!\sqrt{\pi}}{2^n a^{\frac{2n+1}{2}}}$$

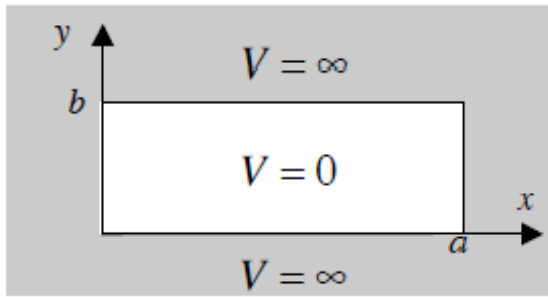
$$\psi_{0,0,0}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}} \exp\left(-\frac{r}{a_0}\right); a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$$

$$m = 9.1\times10^{-31}\text{kg},\ \hbar = 1.05\times10^{-34}\text{Js},\ \epsilon_0 = 8.8\times10^{-12}\ \frac{\text{C}^2}{\text{Nm}^2}, e = 1.6\times10^{-19}\text{C}$$

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z$$

$$\begin{aligned}\hat{x} &= \sqrt{\frac{\hbar}{2m\omega_0}}\left(b + b^\dagger\right)\\ \hat{p} &= -i\sqrt{\frac{\hbar m\omega_0}{2}}\left(b - b^\dagger\right)\end{aligned}$$

P1. A particle with mass m moving in two-dimensions is confined to an infinite rectangular-well potential having length a and width b. What are the energy eigenfunctions and associated eigenvalues of the system? Under what conditions are there degeneracies in the spectrum?



P2. A harmonic oscillator with frequency ω_0 is in the ground state. At time t_0 the frequency is abruptly reduced by a factor of two. For $t > t_0$, what is the probability that the new oscillator, with frequency $\omega_0/2$, will be in its ground state?

P3. Calculate the root-mean-square (RMS) speed of the electron in a hydrogen atom in the ground state (in m/s).

P4. An electron beam is prepared so that each particle is in the spin state $|\psi\rangle = \frac{1}{\sqrt{3}}|\uparrow\rangle + i\sqrt{\frac{2}{3}}|\downarrow\rangle$, where the states $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of $\hat{S}_z$. Along an axis z' a measurement of $\hat{S}_{z'}$ is made, and $S_{z'} = +\hbar/2$ is always found. ($\hat{S}_{z'} = -\hbar/2$ is never found.) What is the angle θ between the z' axis and the z axis?

P5. A particle's wave function is

$$\psi(x, y, z) = C(xy + 2yz)e^{-\alpha r}$$

where C and α are constants. What are the possible outcomes if L^2 is measured? What are the possible outcomes if L_z is measured? You may need the following spherical harmonics:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8}}\sin\theta e^{\pm i\varphi} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_2^{\pm 2} = \mp\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi} \quad Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\varphi} \quad Y_2^0 = \mp\sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

P6. A free electron is initially prepared in a Gaussian wave packet,

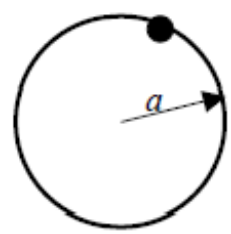
$$\psi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp(-\frac{x^2}{4\sigma^2})$$

with standard deviation $\sigma = 10$ μm in the x direction. Estimate the time (in seconds) for the uncertainty in the electron's position in x to increase to 20 μm.

P7. An electron with mass M is constrained to move on a ring of radius a in the xy plane. A uniform magnetic field B passes through the ring in the z direction. The Schroedinger equation is

$$\frac{1}{2M}\left(\frac{\hbar}{ia}\frac{\partial}{\partial\phi} - \frac{eBa}{2}\right)^2 \psi(\phi) = E\psi(\phi).$$

What are the energy eigenvalues E? Sketch a graph of the ground state energy as a function of B.



P8. A harmonic oscillator with frequency ω_0 and mass m is prepared in the state

$$|\psi\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

where α is a complex number, and $|n\rangle$ is an eigenstate of the number operator $\hat{n} = b^\dagger b$, where $b^\dagger$ and b are the raising and lowering operators. Show that $|\psi\rangle$ is an eigenstate of b, and find the associated eigenvalue. Find $\langle x(t)\rangle$.

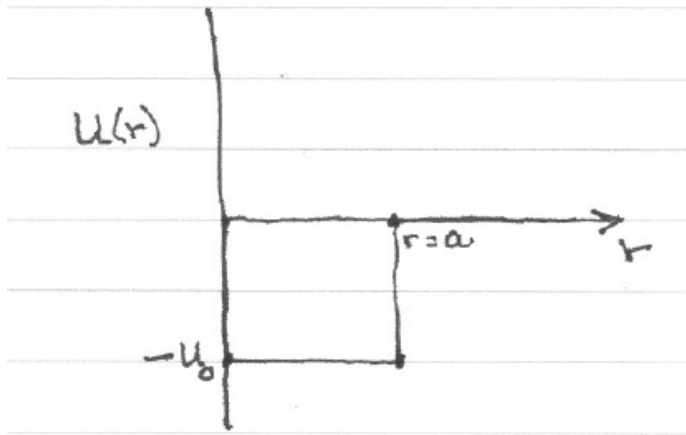
P9. Consider a particle with mass m in a three-dimensional spherical "square" well centered at the origin, described by the potential energy function,

$$U(r) = \begin{cases} -U_0; r < a \\ 0; r \geq a \end{cases},$$

sketched in the figure below. The radial Schroedinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}\Phi + \left(\frac{\hbar^2 \ell(\ell+1)}{2mr^2} + U(r)\right)\Phi = E\Phi$$

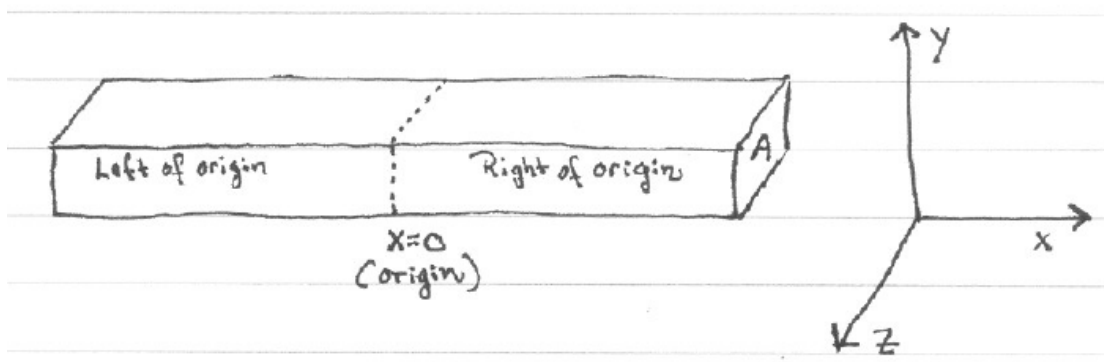
where ℓ is the angular momentum quantum number, and $\Phi(r) = r\Psi(r)$. What is the minimum depth U_0 so that there is at least one bound state?



P10. At $t = 0$, the wave function for a particle with mass m moving in the x direction through a channel with cross sectional area A is

$$\psi(x, y, z) \simeq \frac{1}{\sqrt{AL}} \left(\sin \frac{\pi x}{a} + i \sin \frac{2\pi x}{a} \right)$$

in the neighborhood of $x = 0$. Here L and a are constants. Use the probability flux associated with $\psi(x, y, z)$ to find the time rate of change in the probability to find the particle to the left of the origin $x = 0$, at $t = 0$.



Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy

University of New Mexico

Fall 2014

Instructions:

- The exam consists of 10 problems (10 points each).
- Where possible, show all work; partial credit will be given if merited.
- Useful formulae are provided below; crib sheets are not allowed.

Useful Formulae:

$$E = \left(n + \frac{1}{2}\right)\hbar\omega \ ; \ \hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}}\left(b + b^\dagger\right) \ ; \ \hat{p} = -i\sqrt{\frac{\hbar m\omega_0}{2}}\left(b - b^\dagger\right)$$

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{1}{\pi x_0^2}\right)^{1/4} e^{-\frac{x^2}{2x_0^2}} H_n\left(x / x_0\right); \ x_0 = \sqrt{\frac{\hbar}{m\omega}} \ ; \ H_n(x) = \left(-1\right)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ D(\hat{n}, \theta) = \exp\left(-i\frac{\theta}{\hbar}\hat{n}\cdot\vec{S}\right)$$

States of total spin J for two spin-1 particles:

$$\left|J = 2, M = 2\right\rangle = \left|m_1 = 1, \mathrm{m}_2 = 1\right\rangle$$

$$\left|J = 2, M = 1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|1,0\right\rangle + \left|0,1\right\rangle\right)$$

$$\left|J = 2, M = 0\right\rangle = \frac{1}{\sqrt{6}}\left(\left|1,-1\right\rangle + 2\left|0,0\right\rangle + \left|-1,1\right\rangle\right)$$

$$\left|J = 2, M = -1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|-1,0\right\rangle + \left|0,-1\right\rangle\right)$$

$$\left|J = 2, M = -2\right\rangle = \left|m_1 = -1, \mathrm{m}_2 = -1\right\rangle$$

$$\left|J = 1, M = 1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|1,0\right\rangle - \left|0,1\right\rangle\right)$$

$$\left|J = 1, M = 0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|1,-1\right\rangle - \left|-1,1\right\rangle\right)$$

$$\left|J = 1, M = -1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|-1,0\right\rangle - \left|0,-1\right\rangle\right)$$

$$\left|J = 0, M = 0\right\rangle = \frac{1}{\sqrt{3}}\left(\left|1,-1\right\rangle - \left|0,0\right\rangle + \left|-1,1\right\rangle\right)$$

P1. At time $t=0$ a particle with mass m is incident from the left $(x<0)$ on a step potential,

$$V(x)=V_0,\quad x>0$$
$$V(x)=0,\quad x<0$$

If the particle has energy $E=\frac{4}{3}V_0$, what is the probability that it will be on the right (x>0) after a long time ($t\to\infty$)?

P2. Consider a particle with mass m in a one-dimensional quadratic potential given by $U(x)=\frac{1}{2}kx^2$, $(k>0)$, in the state

$$\psi(x)=\left(\frac{2a}{\pi}\right)^{1/4}\exp(-ax^2),\quad -\infty<x<\infty$$

Evaluate the expectation value of the energy of the particle in this state, and find the value of the parameter a for which this energy is a minimum. Useful integral:

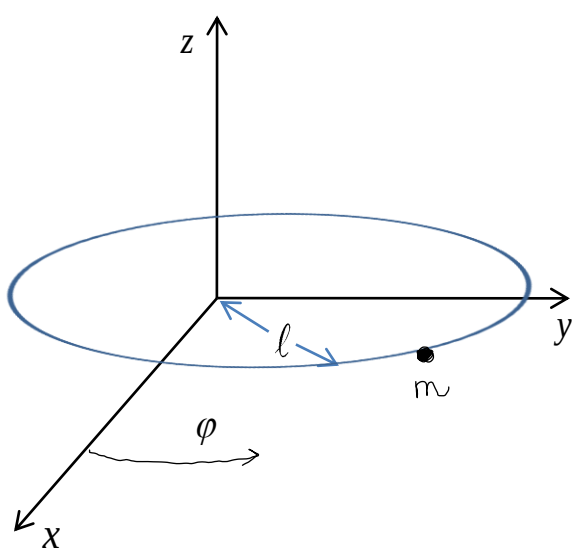
$$\int_{-\infty}^{\infty} dx x^{2n}\exp\left(-\beta x^2\right)=\sqrt{\pi}\left(-1\right)^n\frac{d^n}{d\beta^n}\left(\beta^{-1/2}\right)$$

P3. Consider a two-state system for which the Hamiltonian is

$$\hat{H}=V\left(|1\rangle\langle 2|+|2\rangle\langle 1|\right)+\Delta\left(|1\rangle\langle 1|-|2\rangle\langle 2|\right)\ ,$$

where V and Δ are real. At time $t=0$ the system is in the state $|1\rangle$. What is the probability that the system will be in the same state at a later time t?

P4. A particle with mass m is constrained to move along the perimeter of a circle of radius ℓ in the xy plane. Write down a Hamiltonian H and find the eigenvalues and associated eigenfunctions in terms of the azimuthal angle φ . Find another observable that distinguishes between the degenerate states of H, and obtain the eigenfunctions and eigenvalues of the operator associated with this observable.



P5. Consider a particle with mass *m* moving in one dimension, its dynamics described by the harmonic oscillator Hamiltonian, $\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega_0^2 \hat{x}^2$. At time t = 0 the state is prepared so that the expectation values of position and momentum are, respectively, $\langle \hat{x} \rangle = x_0$ and $\langle \hat{p} \rangle = p_0$. Derive expressions for $\langle \hat{x}(t) \rangle$ and $\langle \hat{p}(t) \rangle$ in terms of x_0 and p_0.

P6. A particle with mass m moves about the origin in a three-dimensional spherically symmetric potential $V(r)$. Its dynamics is determined by the Hamiltonian $\hat{H} = \frac{1}{2m}\left(\hat{P}_r^2 + \frac{\hat{L}^2}{r^2} \right) + V(r)$, where, in spherical coordinates, $\hat{P}_r = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r} r$ and $\hat{L}^2 = -\hbar^2 \left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} \right) \right)$. Consider finding the eigenfunctions of $\hat{H}$ using the method of separation of variables. Taking the ansatz that an eigenfunction $\psi(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi)$ can be written as a product of three functions, one for each coordinate, show that the Schroedinger equation can be separated into three independent eigenvalue problems, one for each of the functions $R, \Theta,$ and Φ, respectively. [Note: φ is the azimuthal angle, determining longitude.]

P7. A photon traveling in the positive z direction is in the state

$$|\psi\rangle = \frac{1}{\sqrt{10}}\left(|R\rangle - 3|L\rangle \right)$$

where $|R\rangle$ and $|L\rangle$ refer to right and left circularly polarized states, respectively. What is the probability for the photon to be polarized along the x axis?

P8. Two spin-1 bosons are confined to a harmonic oscillator potential in the *x* direction and are subjected to a uniform magnetic field, $\vec{B} = B_0\hat{z}$. Ignoring particle-particle interactions and orbital magnetic coupling, the Hamiltonian is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{\mu}{\hbar}\left(\vec{S}_1 + \vec{S}_2 \right) \cdot \vec{B},$$

where the subscripts 1 and 2 refer to particles 1 and 2, respectively. By appropriately combining harmonic oscillator eigenfunctions and two-particle spinnors (consult the formula table on page 1), write down explicit expressions for the two-body wave functions and their associated energies for the ground state and the first excited state(s). Include both space and spin degrees of freedom. Assume that $\mu B_0 << \hbar\omega_0$.

P9. An electron with mass m moving in one-dimension is confined by an infinite square-well potential to the region $-L/2 < x < L/2$. The eigenfunctions for the time-independent Schroedinger equation may be written in two different ways,

$$\psi_n(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right),$$

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right),$$

depending on whether the integer n is even or odd. Suppose that the electron is perturbed with an electric field E, giving rise to a perturbing potential $V(x) = eEx$. To lowest non-vanishing order in E, what is the shift in the ground state energy? Assume that the effect of E can be adequately described with a truncated basis set consisting of only the two lowest-energy eigenfunctions.

[Useful Integral: $\int_{-\pi}^{\pi} \sin(\varphi)\cos(\varphi/2)\varphi d\varphi = \frac{32}{9}$]

P10. An electron is prepared in the spin-up state with respect to the z axis. Consider a second axis z' that is titled from the z axis by an angle of 105^0. What is the probability that a measurement of the spin along the z' axis will give a value of $\hbar/2$?

Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy
University of New Mexico
Summer 2015

Instructions:

- The exam consists of 10 problems (10 points each).
- Where possible, show all work; partial credit will be given if merited.
- Useful formulae are provided below; crib sheets are not allowed.
- Total time: 3 hours

Useful Information:

$$E = \left(n + \frac{1}{2}\right)\hbar\omega_0 \;;\; \hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}}\left(b + b^\dagger\right) \;;\; \hat{p} = -i\sqrt{\frac{\hbar m\omega_0}{2}}\left(b - b^\dagger\right)$$

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{1}{\pi x_0^2}\right)^{1/4} e^{-\frac{x^2}{2x_0^2}} H_n\left(x/x_0\right);\; x_0 = \sqrt{\frac{\hbar}{m\omega}} \;;\; H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Table 1: Wave functions and their components

n	ℓ	m	$R_{n\ell}$	$Y_{\ell m}$	$\psi_{n\ell m} = R_{n\ell} Y_{\ell m}$
1	0	0	$2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$
2	0	0	$\left(\frac{1}{2a_0}\right)^{3/2}\left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2}\left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$
2	1	0	$\left(\frac{1}{2a_0}\right)^{3/2}\frac{1}{\sqrt{3}}\frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$	$\frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2}\frac{r}{a_0} e^{-r/2a_0}\cos\theta$
2	1	± 1	$\left(\frac{1}{2a_0}\right)^{3/2}\frac{1}{\sqrt{3}}\frac{r}{a_0} e^{-r/2a_0}$	$\pm\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{\pm i\phi}$	$\frac{1}{8}\sqrt{\frac{1}{\pi}}\left(\frac{1}{a_0}\right)^{3/2}\frac{r}{a_0} e^{-r/2a_0}\sin\theta e^{\pm i\phi}$

$$E_{n,\ell,m} = -\frac{\alpha^2 mc^2}{2n^2}$$

$$1\text{eV} = 1.6\times 10^{-19}\,\text{J}$$

$$\left(4\pi\varepsilon_0\right)^{-1} = 9\times 10^9\,\text{Nm}^2/\text{C}^2; \hbar = \left(2\pi\right)^{-1}\times 6.626\times 10^{-34}\,\text{Js}$$

$$a_0 = 0.529\,\overset{0}{\text{A}};\; 1\text{eV} = 1.6\times 10^{-19}\,\text{J};\; \frac{e^2}{4\pi\varepsilon_0 a_0} = 27.2\text{eV}$$

$$\mu_B = \frac{e\hbar}{2m_e}$$

$$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}; \hbar c \simeq 197\,\text{eV-nm}; \text{m}_e \simeq 0.511\,\text{MeV/c}^2;\ \text{m}_p \simeq 938\,\text{MeV/c}^2$$

$$\int_{-\infty}^{\infty} dx\, x^{2n} \exp(-ax^2) = \frac{(2n-1)!!}{2^n}\sqrt{\frac{\pi}{a^{2n+1}}}$$

$$\int_{-\infty}^{\infty} dx \exp\left(-ax^2\right)\exp\left(ikx\right) = \sqrt{\frac{\pi}{a}}\exp\left(-\frac{k^2}{4a}\right)$$

$$\int_{-\infty}^{\infty} dx\left(x^2+a^2\right)^{-1} = \frac{\pi}{a}\exp\left(-a|k|\right)$$

$$\int_0^{\infty} x^s dx \exp\left(-ax\right) = \frac{\Gamma(s+1)}{a^{s+1}}; \Gamma\left(s+1\right) = s\Gamma\left(s\right); \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}; \Gamma\left(n+1\right) = n!$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z\ ,\ \hat{D}(\vec{n},\theta) = \exp\left(-i\frac{\theta}{\hbar}\vec{n}\cdot\hat{\vec{S}}\right)$$

P1. An electron with spin initially in an eigenstate of $\hat{S}_x$ with an eigenvalue $\hbar/2$ is placed in a uniform magnetic field $\vec{B} = B_0\hat{z}$. Show that the expectation value $\left\langle \hat{S}_x(t)\right\rangle$ is a periodic function of time, and calculate its period (in seconds) for the case that B_0 = 1.0 Tesla.

P2. The Hamiltonian for a harmonic oscillator is given by

$$\hat{H} = \hbar\omega_0\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right)$$

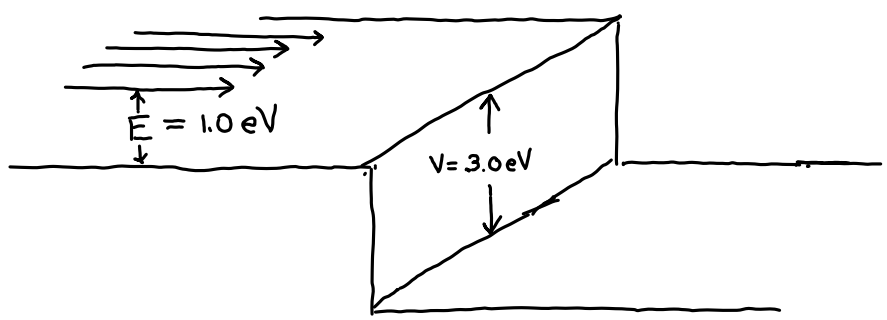
where the operators $\hat{b}$ and $\hat{b}^\dagger$ satisfy the commutation relation $\left[\hat{b}, \hat{b}^\dagger\right] = 1$. Consider an eigenstate $|n\rangle$ of $\hat{H}$ with eigenvalue $\left(n+\frac{1}{2}\right)\hbar\omega_0$, where n is a positive integer. Prove that the two states, $\left|\phi_{(-)}\right\rangle = C_{(-)}\hat{b}|n\rangle$, and $\left|\phi_{(+)}\right\rangle = C_{(+)}\hat{b}^\dagger|n\rangle$, are also eigenstates of $\hat{H}$, where $C_{(-)}$ and $C_{(+)}$ are normalization constants. Find their respective eigenvalues and normalization constants.

P3. If the photon had a non-vanishing mass m^*, electrons and protons would attract one another via a screened (Yukawa) potential of the form

$$U(r) = \frac{-e^2}{4\pi\varepsilon_0 r} e^{-\gamma r} \ ,$$

where the screening parameter $\gamma = m^* c / \hbar$. Such an effect would cause a shift in the hydrogen emission spectrum. *To lowest non-vanishing order in* γ, what would be the energy (eV) shift in the Lyman series 2p-to-1s spectral line if the photon mass were as large as the $0.320\,\text{eV/c}^2$ neutrino mass? (Note: The energy associated with a spectral line is the energy *difference* between initial state and final atomic states. As such, a shift in the position of a spectral line will only occur if the initial and final state energies are each perturbed by different amounts.)

P4. A nonrelativistic beam of particles with kinetic energy E=1.0 eV is normally incident on a fall-away 3.0 eV potential step. A two-dimensional slice of the potential step with energy on the vertical axis is sketched for reference below. If the particle number density in the incoming beam is $1.0 \times 10^{-3}\,\text{m}^{-3}$, what will be the number density in the transmitted beam?



P5. A 5 eV electron moves freely along the x axis. In momentum-space the electron wave function is a Gaussian, peaked at momentum $\hbar k_0$. In real-space the wave function evolves in time according to

$$\psi(x,t) = A\exp\left[i\left(k_0 x - \omega t\right)\right]\exp\left[-\frac{\left(x-\omega' t\right)^2}{4\left(\sigma^2 + \frac{i}{2}\omega'' t\right)}\right] ,$$

where A and σ are constants and $\omega' = \left(\partial\omega/\partial k\right)\big|_{k=k_0}$, $\omega'' = \left(\partial^2\omega/\partial k^2\right)\big|_{k=k_0}$ are the first and second derivatives of the dispersion relation $\omega(k)$. If the uncertainty in position initially is $\Delta x = 1.0\,\text{mm}$, after how much time (in seconds) will $\Delta x = 10.0\ \text{mm}$?

P6. The nuclei of many large atoms are found to be aspherical, their rotational motion described by the Hamiltonian for an axially symmetric rotator,

$$H = \frac{L_x^2 + L_y^2}{2I_1} + \frac{L_z^2}{2I_2}$$

How much smaller than I_1 does I_2 have to be for the energy levels of the $\ell = 1$ multiplet to begin to overlap the energy levels of the $\ell = 2$ multiplet?

P7. A free particle with mass m is subjected to a spatially-uniform sinusoidal driving force,

$$\vec{F}(t) = \vec{F}_0 \sin(\omega t) \ ,$$

with frequency ω and amplitude $\vec{F}_0$ directed along the x-axis. Obtain an expression for the expectation value of the position $\langle \hat{x}(t) \rangle$ as a function of time in terms of the expectation values of the initial position $\langle \hat{x}(0) \rangle$ and the initial momentum $\langle \hat{p}(0) \rangle$.

P8. The Hamiltonian for an anharmonic oscillator in one dimension is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + g\hat{x}^4$$

where g is a positive constant. The ground state energy E can be estimated using the variational method. For a trial wave function, $\psi(x) = \left(\frac{2\alpha}{\pi}\right)^{1/4} \exp\left(-\alpha x^2\right)$, it is found that

$$\langle \psi | \hat{H} | \psi \rangle = \frac{\hbar^2 \alpha}{2m} + \frac{3}{16}\frac{g}{\alpha^2}$$

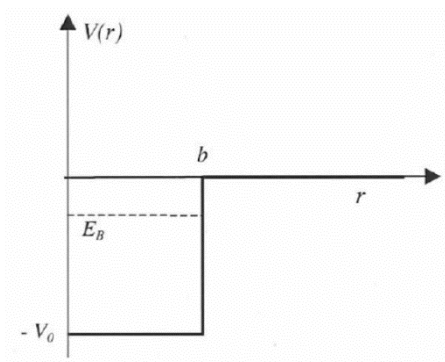
Show that this implies,

$$E \geq \left(\frac{3}{4}\right)^{4/3} \frac{\hbar^{4/3} g^{1/3}}{m^{2/3}} \ .$$

P9. A deuteron is a bound state of a proton and a neutron, $m_N \simeq m_P \simeq 939\ \mathrm{MeV/c^2}$. The attraction between the two particles can be modeled by a spherical square-well potential of width b and depth $-V_0$, as illustrated in the figure. The reduced radial wavefunction $u(r) = rR(r)$ for the ground state is determined from the time-independent Schroedinger equation for motion relative to the center of mass,

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} + V(r)u = Eu$$

Assuming a bound state exists, sketch the shape of the ground state wave function. For a width $b = 1.63\ \mathrm{fm}$ ($1\mathrm{fm} = 10^{-15}\mathrm{m}$) calculate the threshold depth V_0 (in MeV) required to have at least one bound state.



P10. The wave function for a particle moving freely in three dimensions is given by

$$\psi(x,y,z) = \frac{1}{\left(2\pi\sigma^2\right)^{3/4}} \exp\left(-\frac{x^2+y^2+z^2}{4\sigma^2}\right)$$

where σ is a constant. Calculate the distribution of expected outcomes of a measurement of the particle's energy E. What is the mean of this distribution?

Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy
University of New Mexico

Spring 2005

Instructions:

• The exam consists of two parts: 5 short answers (6 points each) and your choice of 2 out 3 long answer problems (35 points each).
• Where possible, show all work, partial credit will be given.
• Personal notes on two sides of a 8X11 page are allowed.
• Total time: 3 hours

Good luck!

Short answers:

S1. Consider a free particle described by a Gaussian wavepacket of width Δx_0. Estimate the time it would take for the width to double. Explain your reasoning.

S2. Estimate the speed of the electron around the nucleus in the ground state of hydrogen. Express its ratio with the speed of light in terms of the fine structure constant. How good of approximation is it to use nonrelativistic theory?

S3. A particle of mass M moves on a ring of radius *a*. Show that the energies are quantized as $E_m = m^2 \frac{\hbar^2}{2Ma^2}$, where $m = 0, \pm 1, \pm 2, \ldots$ Explain the degeneracies.

S4. A one dimensional simple harmonic oscillator is prepared at *t*=0 in the state $|\psi\rangle = \frac{1}{2}|1\rangle + i\sqrt{\frac{3}{4}}|2\rangle$, where $|n\rangle$ is the eigenstate with energy $(n+1/2)\hbar\omega$. What is the state at a later time? What is the probability distribution of possible *n*? What is the mean excitation number $\langle n \rangle$ as a function of time?

S5. Consider a particle of mass *m* moving in 3 dimensions. Which of the following observables can be measured simultaneously:

$$\hat{A} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x,\ \hat{B} = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y,\ \hat{C} = \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = \hat{r}^2,$$

where $(\hat{x}, \hat{y}, \hat{z})$ and $(\hat{p}_x, \hat{p}_y, \hat{p}_z)$ are the three components of position and momentum, respectively, and $\hat{r}$ is the radius from the origin.

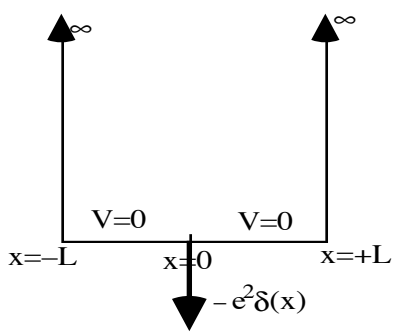
L1. A one dimensional model of Hydrogen

A very schematic **one dimensional model** of a hydrogen atom is given by treating the Coulomb attraction of the electron to the proton by a delta function. With nucleus at the origin, the potential for the electron is $V(x) = -e^2\delta(x)$ (in cgs units), where e is the magnitude of electron and proton charges.

(a) Using the boundary conditions that the wave function is everywhere continuous, but its derivative at the delta function is discontinuous, $\left.\frac{d\psi}{dx}\right|_{0_-} - \left.\frac{d\psi}{dx}\right|_{0_+} = \frac{2me^2}{\hbar^2}\psi(0)$, show that there is one bound state, with eigenfunction $\psi(x) = \sqrt{\kappa}e^{-\kappa|x|}$, with $\kappa = \frac{me^2}{\hbar^2}$, corresponding to bound-state energy $E_b = -\frac{me^4}{2\hbar^2}$.

(b) How does the binding energy compare with the binding energy in a real 3D hydrogen atom in its ground state? How does the rms width of the wave function compare with the Bohr radius $a_0 = \frac{\hbar^2}{me^2}$? (Given: $\int dx\, x^2 e^{ax} = e^{ax}(2 + 2ax + a^2x^2)/a^3$)

(c) Now consider the atom at the origin, inside a box of width 2L, having infinite walls, and located symmetrically with respect to the origin.
Let $V(x) = 0$ inside the box, except at the origin, where $V(x) = -e^2\delta(x)$



Show that the equation determining the binding energy of the ground state is

$$\kappa a_0 = \tanh(\kappa L), \text{ with energy eigenvalue } E = -\frac{\hbar^2\kappa^2}{2m}.$$

Sketch the wave function in this case.

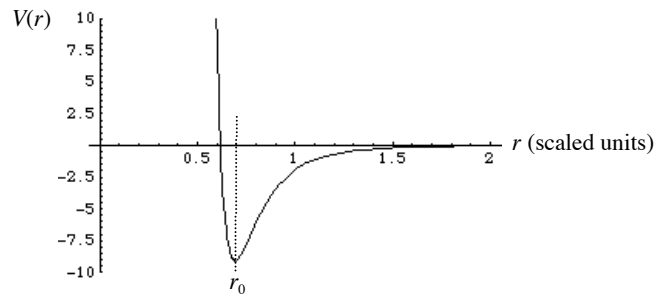
(d) With the proton at the center of the box, determine the *smallest* value of L/a_0, that permits the electron to remain bound to the nucleus (*Hint*: solve graphically).

L2. Diatomic Molecule: Anharmonic Oscillator

A model for the binding potential of two atoms into a molecule by a central force is the so-called "Lennard-Jones" potential,

$$V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$$

where r is the distance between atoms and A and B are positive coefficients



The two-atom energy eigenstates separates into center-of-mass and relative coordinates. The relative coordinate further separates into radial and angular coordinates, as,

$$\psi_{n,l,m}(r,\theta,\phi) = \frac{u_{nl}(r)}{r} Y_{lm}(\theta,\phi),$$

where $Y_{lm}(\theta,\phi)$ is a spherical harmonic. The (reduced) radial wave function satisfies,

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right) u_{nl}(r) = E u_{nl}(r),$$ where m is the reduced mass.

(a) What is the meaning of each of the terms in this Schrödinger equation?
(b) Show that ***classically***, for l=0, the equilibrium separation between the atoms is $r_0 = (2A/B)^{1/6}$, and the vibrational frequency about equilibrium is $\omega = \sqrt{V_0''/m}$ where $V_0'' = \frac{d^2V}{dr^2}(r_0)$.
(c) If you approximate the motion near the ground state as a harmonic oscillator, what is the ground state energy for l=0?
(d) Now included the first anharmonic correction to the l=0 potential. To lowest non-vanishing order in perturbation theory, show that the shift in the ground-state energy is,

$$\delta E_{ground} = -\xi^2 \left(\frac{\hbar}{2m\omega}\right)^3 \sum_{n\neq 0} \frac{\left|\langle n|\left(\hat{a}+\hat{a}^\dagger\right)^3|0\rangle\right|^2}{n\hbar\omega},$$

where $\xi = \frac{1}{6}\frac{d^3V}{dr^3}\bigg|_{r_0}$, $|n\rangle$ are the eigenstates of the harmonic oscillator, and $\hat{a}^\dagger / \hat{a}$ are the usual raising/lowing operators.

L3. The hyperfine structure of hydrogen

The ground electronic state of hydrogen has "hyperfine structure" – a splitting due of the spectral lines due to the interaction of the spins of the electron and proton.

(a) Including electron and nuclear spins, but neglecting their interaction, write the quantum numbers associated with the four degenerate sublevels of the hydrogen ground state.

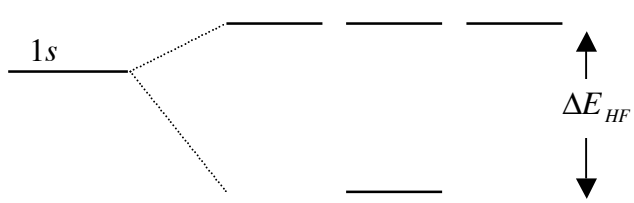
Now lets add the hyperfine interaction. The interaction Hamiltonian has the form,

$$\hat{H}_{\text{int}} = A\delta^{(3)}(\mathbf{r})\,\hat{\vec{s}}\cdot\hat{\vec{i}}$$

where A is a constant depending on the magnetons, $\hat{\vec{s}}$ is the electron's spin-1/2 angular momentum and $\hat{\vec{i}}$ is the proton's spin-1/2 angular momentum.

(b) Let $\vec{f} = \vec{i} + \vec{s}$ be the total spin angular momentum. Show that ground state manifold are defined with good quantum numbers $\left|n = 1, l = 0, s = 1/2, i = 1/2, f, m_f\right\rangle$. What are the possible values of f and m_f?

(c) The perturbed $1s$ ground state now has hyperfine splitting. The energy level diagram is sketched below. Label all the quantum numbers for the four sublevels shown.



(d) Show that the energy level splitting is $\Delta E_{HF} = A\left|\psi_{1s}(0)\right|^2$ where $\psi_{1s}(0)$ is the value of the $1s$ electron orbital at the position of nucleus.

This splitting gives rises to the famous 21cm radio frequency radiation used in astrophysical observations.

<u>Preliminary Examination: Quantum Mechanics</u>

Department of Physics and Astronomy

University of New Mexico

Spring 2006

Instructions:

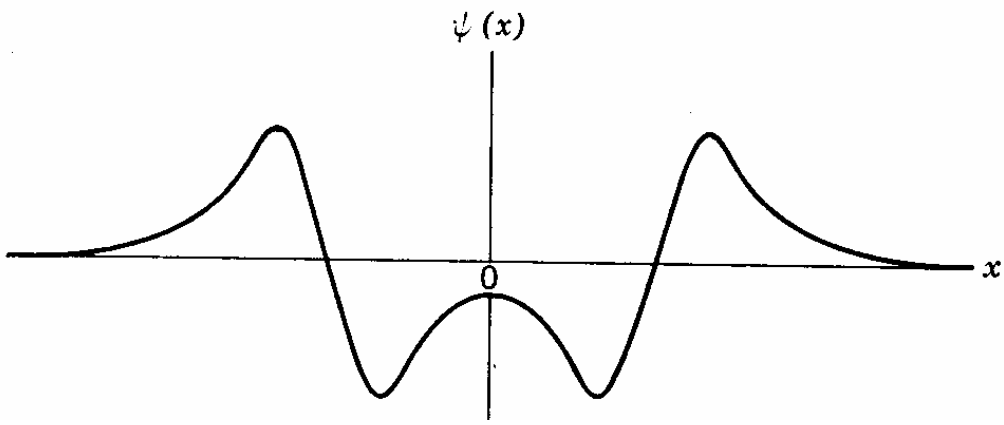
- The exam consists of 10 short answers (10 points each).
- Where possible, show all work, partial credit will be given.
- Personal notes on two sides of an 8X11 page are allowed.
- Total time: 3 hours

Good luck!

1. A particle of mass *m* moves in a potential: $V(x) = \frac{1}{2}kx^2 + cx$. Find the eigenvalue of the nth state (n= 0,1,2,…) to lowest *nonvanishing* order in perturbation theory, treating the linear term *cx* as a perturbation.

2. A particle of mass *m* is in the ground state of an infinite square well potential, $V(x) = \begin{cases} 0, 0 \le x \le a \\ \infty, \text{otherwise} \end{cases}$. Suddenly, the well expands to three times its original size – the right wall moving from *a* to *3a* – leaving the wave function momentarily undisturbed. The energy of the particle is now measured. Write an integral expression for the probability of finding the particle in the ground state of the new potential.

3. Given the eigenfunction below, make a sketch of the potential (on the same graph), indicating on the graph the energy level associated with this eigenfunction. Describe the energy of this eigenfunction relative to the lowest allowed energy.



4. Quarks carry spin ½. Three quarks bind together to make a baryon (such as the proton or neutron); two quarks (or more precisely a quark and an antiquark) bind together to make a meson (such as the pion or kaon). Assume the quarks are in the ground state (so the orbital angular momentum is zero). What magnitudes of the spin vector are possible for baryons and mesons?

5. In the representation in which S^2 and S_z are both diagonal, a spin- ½ wave function is:

$$\psi = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3i \end{pmatrix}.$$

What is the probability that a measurement of S_x on ψ yields $-\hbar/2$?

6. A particle with zero spin in a spherically symmetric potential is in a state represented by the wavefunction:

$\psi(x,y,z) = C(xy + 2yz)e^{-\alpha r}$, where C and α are constants.

a) What is the eigenvalue of L^2 for this particle?
b) If L_z were measured, what are the possible outcomes?

You may need the following spherical harmonics:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8}}\sin\theta e^{\pm i\varphi} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_2^{\pm 2} = \mp\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi} \quad Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\varphi} \quad Y_2^0 = \mp\sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

7. Operators $\hat{A}$ and $\hat{B}$ obey the following equations: $\hat{A}\psi_{ab} = a\psi_{ab}$ and $\hat{B}\psi_{ab} = b\psi_{ab}$, where the ψ_{ab} form a complete set in Hilbert space and *a* and *b* are real numbers. Show that the operator $\left(\hat{A}\hat{B}\right)$ is Hermitian.

8. A particle in a finite square well potential:

$$V(x) = \begin{cases} 0, & x < -a \\ -V_0, & -a \le x \le a \\ 0, & x > a \end{cases},$$

with only three bound states, initially is in a state such that the probability to measure the energy is given by: $P(E_1) = 1/3$, $P(E_2) = 1/3$ and $P(E_3) = 1/3$. The parity is then measured in such a way that the particle stays bound, and found to be -1. If some time later, the energy is measured, what value is found?

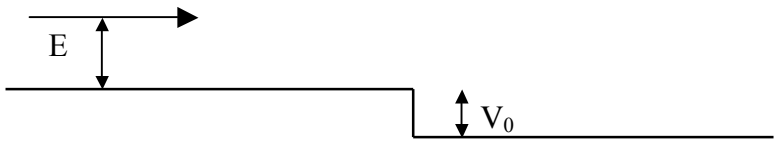
9. In a photoelectric experiment, electrons are emitted from a surface illuminated by light of wavelength 4000Å, and the stopping potential for these electrons is found to be Φ_0 = 0.5V. What is the longest wavelength of light that can illuminate this surface and still produce a photoelectric current?

10. A free electron is prepared in a Gaussian wave packet with standard deviation σ_x. Estimate the time for the uncertainty in the electron's position to increase to 2 σ_x.

Preliminary Examination: Quantum Mechanics
Department of Physics and Astronomy
University of New Mexico
Spring 2007

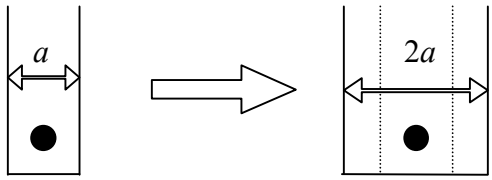
Instructions:
- The exam consists of 10 problems, 10 points each.
- Partial credit will be given if merited.
- Personal notes on two sides of an 8 × 11 page are allowed.
- Total time is 3 hours.

Problem 1: A nearly monoenergic particle is moving in one dimension where the potential energy make a sharp jump as shown.

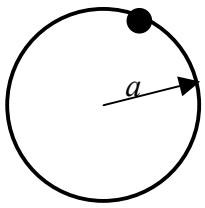


What is the probability for the particle to transmit across the step in according to classical mechanics and according to quantum mechanics?

Problem 2: A particle of mass *m* is trapped in one dimension in a box with hard walls (infinite energy to escape) of width *a*. At time *t*=0, the particle is in the ground state. The width of the box is *suddenly* expanded to 2a. What is the probability of finding the particle in the ground state of the new well (you may leave your answer as an integral)? What is the probability of finding the particle in the first excited state of the new well (give a number).



Problem 3: A particle of mass *m* moves in two dimensions constrained to a ring of radius *a*. The particle is otherwise "free" (no other external forces act on the particle). What are the conserved physical quantities? What are energy eigenvalues and degeneracies of these levels? How are the degeneracies related to the conserved quantities?



Problem 4: A wave packet in a 1D harmonic oscillator is described by the state,

$$|\psi\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$

where α is a complex number and $|n\rangle$ is an eigenstate of the number operator, $\hat{N} \equiv \hat{a}^\dagger \hat{a}$, where $\hat{a}^\dagger, \hat{a}$ are the usual creation and annihilation operators.

(i) Show that this state is normalized?
(ii) Is this an energy eigenstate?
(iii) Show that $\hat{a}|\psi\rangle = \alpha|\psi\rangle$.

Problem 5: Consider a particle with angular momentum J=1. Show that,

$$|\psi_a\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}\left(|1\rangle + |-1\rangle\right),\ |\psi_b\rangle = -\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}\left(|1\rangle + |-1\rangle\right),\ |\psi_c\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle - |-1\rangle\right)$$

are eigenstates of the x-component of angular momentum, $\hat{J}_x$, with the eigenvalues you expect. Here $\{|m\rangle \mid m = -1,0,1\}$ denote the three eigenstates of $\hat{J}_z$, $\hat{J}_z|m\rangle = m|m\rangle$. (Hint, express $\hat{J}_x$ in terms of raising and lowering operators).

Problem 6: Consider again, a particle with angular momentum J=1, whose dynamics is governed by a Hamiltonian $\hat{H} = \kappa \hat{J}_z^2$. At time t=0, we prepare the particle in the state $|\psi(0)\rangle = |\psi_a\rangle$, as given in Problem 5. Show that at a time $t = \pi/(2\kappa)$,

$$|\psi(t = \pi/(2\kappa)\rangle = \frac{e^{-i\pi/4}|\psi_a\rangle - e^{i\pi/4}|\psi_b\rangle}{\sqrt{2}}.$$

Problem 7: Consider two distinguishable noninteracting particles of mass m, each trapped in a separate harmonic well in 1D. The Hamilton is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2(x_1 + a)^2 + \frac{1}{2}m\omega^2(x_2 - a)^2.$$

(i) Show that this Hamiltonian separable in relative coordinate and center of mass.
(ii) Express the total energy eigenvalues of the two-body system in terms energy eigenvalues associated with the motion of center of mass and relative coordinate.

Problem 8: Neglecting the spin of the electron or proton, what is the degeneracy of the first excited state of a hydrogen atom? What are the quantum numbers that uniquely specify each of the substates? Now add electron spin and spin-orbit coupling. What are the good quantum numbers? Sketch an energy level diagram for each of these levels of description. Calculation of the energy level splitting is not necessary.

Problem 9: Consider the Hamiltonian describing two bound states of an atom. Written in bra-ket notation, the unperturbed Hamiltonian in this subspace is,

$$\hat{H}_0 = E_1|\psi_1\rangle\langle\psi_1| + E_2|\psi_2\rangle\langle\psi_2|.$$

Interaction with a weak external field is described by the Hamiltonian

$$\hat{H}_1 = E_{\text{int}}\left(|\psi_1\rangle\langle\psi_2| + |\psi_2\rangle\langle\psi_1|\right),$$

where $\left|\frac{E_{\text{int}}}{E_2 - E_1}\right| << 1$. To lowest nonvanishing order in perturbation theory, what are the shifts in the energies of the two bound states?

Problem 10: Given the total Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$ in Problem 9, solve for the *exact* energy eigenvalues. Perform a Taylor series expansion on your result to find the perturbation shift in eigenvalues to lowest order in $E_{\text{int}}/(E_2 - E_1)$.

Preliminary Examination: Quantum

Department of Physics and Astronomy
University of New Mexico
Winter 2008

Instructions:

- *the exam consists of 10 problems, 10 points each;*
- *partial credit will be given if merited;*
- *personal notes on two sides of* 8×11 *page are allowed;*
- *total time is 3 hours.*

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1. A spin-1/2 particle is in the state,

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left\{|\uparrow z\rangle + i\sqrt{2}|\downarrow z\rangle\right\},$$

where in this notation $|\uparrow z\rangle$ labels the state that has spin-up with respect to the z-axis. Find the probability for a particle in this state to pass through an Stern-Gerlach device oriented along $+x$ with spin-up followed by a Stern-Gerlach device oriented along $+z$ with spin-down. Note that $|\uparrow x\rangle = \frac{1}{\sqrt{2}}\{|\uparrow z\rangle + |\downarrow z\rangle\}$.

2. A particle is incident from the left ($x < 0$) on step potential,

$$V(x) = V_0, \quad x > 0$$
$$V(x) = 0, \quad x < 0$$

a) For $E < V_0$, sketch the wave function for all x.

b) Let's call the wave function for $x < 0$ Ψ_-. Then for $E < V_0$ Ψ_- must be of the form,

$$\Psi_- = e^{ikx} + e^{i\delta}e^{-ikx},$$

where δ is a real constant. Why?

c) What is δ in the limit $V_0 >> E$?

3. Consider a particle in a 1-D box $0 < x < a$ which at $t = 0$ is in the state,

$$\psi(x) = \sqrt{\frac{3}{a}}\Big(2x/a\Big), \quad 0 < x < a/2$$

$$\psi(x) = \sqrt{\frac{3}{a}}\Big(2 - 2x/a\Big), \quad a/2 < x < a,$$

Determine the probabilities P_1, P_2 to measure the ground state energy E_1 and the first excited state energy E_2 at t=0. (use $\int_0^{\pi/2} u\sin(u)du = 1$)

4. Consider a system with orthonormal basis states $|1\rangle$ and $|2\rangle$. The Hamiltonian in this basis is given by,

$$\hat{H} = \begin{pmatrix} E_0 & -iA \\ iA & E_0 \end{pmatrix}$$

where E_0 and A are real, positive constants. Find the energy eigenvalues and corresponding normalized energy eigenstates. Be clear as to which state goes with which eigenvalue.

5. Consider a spinless particle that is constrained to move on a circle of radius R but free to move around the circle. In a cylindrical coordinate system with the circle in the x-y plane, the wave function depends only on the azimuthal angle ϕ.

a) Write the Hamiltonian for this system.
b) What are the energy eigenvalues and normalized eigenstates? Identify all good quantum numbers and any degeneracies.
c) What is the uncertainty in ϕ for states of definite energy and angular momentum?

6. Consider a particle subject to the harmonic oscillator Hamiltonian,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}.$$

For an initial state of the particle given by a superposition of energy eigenstates,

$$|\Psi(x,0)\rangle = \frac{1}{\sqrt{2}}(|n\rangle + |n+1\rangle).$$

Calculate $\langle x\rangle$ as a function of time.

7. Consider a diatomic molecule as a rigid rotor with moment of inertia I and magnetic moment $\vec{\mu} = -\mu_0\vec{L}$, μ_0 being a postive constant. The molecule is in a uniform magnetic field $\vec{B}$ along the z-axis.

(a) What is the Hamiltonian for this system?

(b) If at $t = 0$ the wave function is,

$$\Psi(0) = \frac{Y_{11} + Y_{10}}{\sqrt{2}},$$

calculate $\langle L_x\rangle$ as a function of time, using

$$\hat{L}_x = \frac{\hat{L}_+ + \hat{L}_-}{2}$$

8. A particle moving in a central potential V(r) has a Hamiltonian,

$$\hat{H} = \frac{1}{2m}\{\hat{P_r}^2 + \frac{\hat{L}^2}{r^2}\} + V(r),$$

where

$$\hat{P_r} = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r}r.$$

a) Show that

$$\Psi(\vec{r}) = \frac{u(r)}{r}Y_{\ell m}(\theta, \phi)$$

satisfies the time independent Schrodinger equation and find the eigenvalue equation for $u(r)$.

b) For an infinite spherical well of radius a, find the ground state and first $\ell = 0$ excited state wave functions (up to a normalization constant) and the corresponding energies.

9. Consider a particle in a 1D box of length a with $0 < x < a$ at the middle of the box. Calculate the first-order correction to the energies for all energy eigenstates due to the purturbation,

$$\hat{H}_{\text{int}} = \alpha\delta(x - a/2)$$

where $\delta()$ is the Dirac delta function and α is a constant.

10. For the $2P_{3/2}$ hydrogen multiplet (states $|n, \ell, j, m_j\rangle$ with $n = 2$, $L = 1$ and $J = 3/2$) calculate the correction due to the Zeeman effect in the weak-field approximation,

$$\hat{H}_B = \frac{eB}{2mc}(\hat{L}_z + 2\hat{S}_z).$$

Note: States of definite total angular momentum $|j, m_j\rangle$ are related to the product states of orbital angular momentum times spin $|\ell, m_\ell\rangle|s, m_s\rangle$ according to:

$|\frac{3}{2}, \pm\frac{3}{2}\rangle = |1, \pm 1\rangle|\frac{1}{2}, \pm\frac{1}{2}\rangle$
$|\frac{3}{2}, +\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|1, 1\rangle|\frac{1}{2}, -\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|1, 0\rangle|\frac{1}{2}, +\frac{1}{2}\rangle$
$|\frac{3}{2}, -\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|1, -1\rangle|\frac{1}{2}, +\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|1, 0\rangle|\frac{1}{2}, -\frac{1}{2}\rangle$

Department of Physics and Astronomy, University of New Mexico

Quantum Mechanics Preliminary Examination

Spring 2009

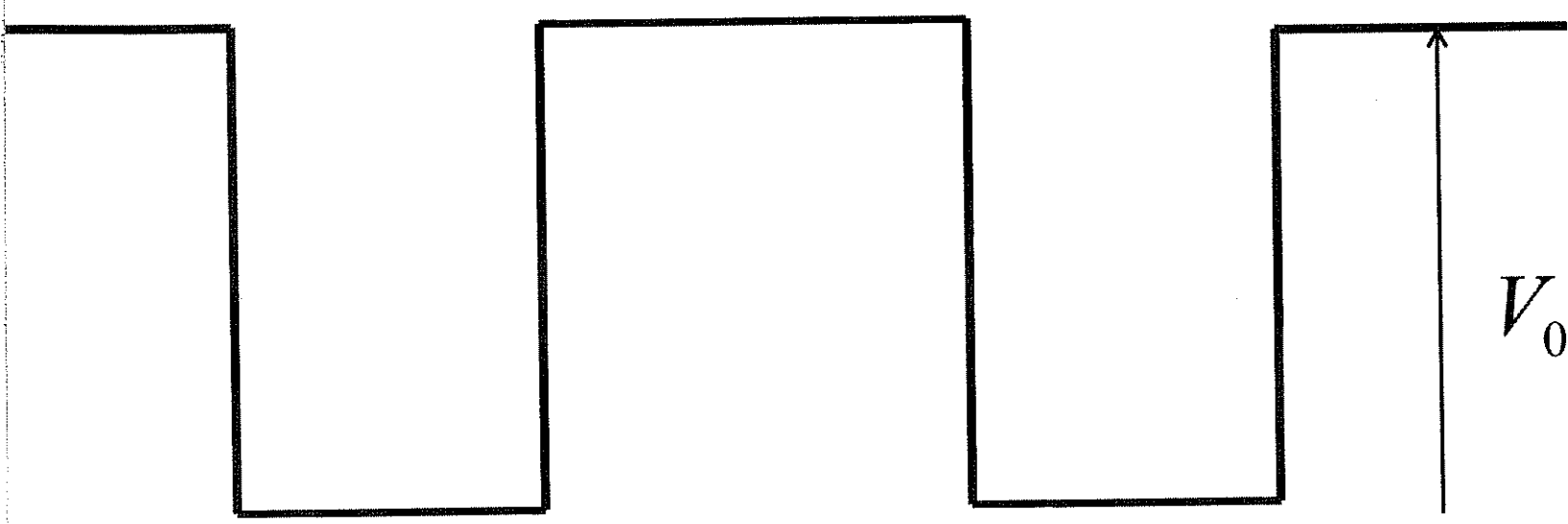
Instructions:

- The exam consists of 10 short-answer problems (10 points each).
- Partial credit will be given if merited.
- Personal notes on two sides of an $8\frac{1}{2}'' \times 11''$ page are allowed.
- Total time: 3 hours.

1– There is a symmetric double well potential as shown in the figure. Assume that this system has at least two bound states, namely the ground state with energy E_0 and the first excited state with energy E_1.

(a) Draw the wavefunctions for the ground state and the first excited state.

(b) What do the wavefunctions look like in the limit that $V_0 \to \infty$? What is the relationship between E_0 and E_1 in this limit?



2– Consider a free particle of mass m in one dimension. At $t = 0$ the wavefunction of the particle in position space is given by the following Gaussian wavepacket:

$$\psi(x) = \frac{1}{\pi^{1/4}\sqrt{a}}\exp[-\frac{(x-x_0)^2}{2a^2}].$$

In momentum space this wavefunction is given by:

$$\psi(p) = \frac{1}{\pi^{1/4}}\sqrt{\frac{2a}{\hbar}}\exp(-\frac{2a^2p^2}{\hbar^2}).$$

(a) Find $\bar{X}$, $\bar{P}$, ΔX, ΔP at $t = 0$ (in terms of a, m and $\hbar$) and show that $\Delta X \Delta P = \hbar/2$. Hint: helpful expression:

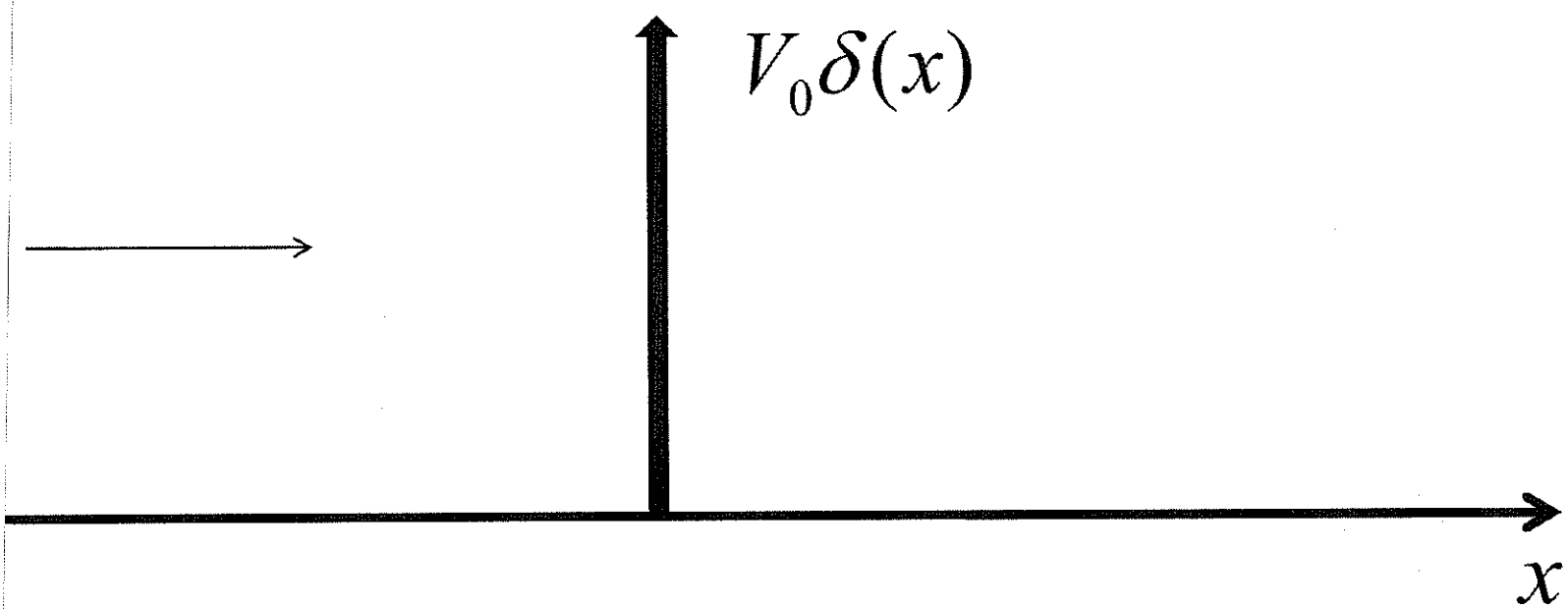
$$\int_0^\infty x^2 e^{-x^2/2} dx = \frac{\sqrt{\pi}}{2}.$$

(b) Describe in words what will happen to $\bar{X}$, $\bar{P}$, ΔX, ΔP at $t > 0$.

3– A particle with mass m moves in one dimension and approaches a localized potential barrier $V = V_0\delta(x)$ from the left as shown in the figure. The wavefunction of the particle is given by (up to an overall normalization factor)

$$\psi(x) = e^{ikx} + Ae^{-ikx} \qquad x \leq 0$$
$$\psi(x) = Be^{ikx} \qquad x > 0\,.$$

Find A and B in terms of V_0 and k.



4– The phenomenon of neutrino oscillations provides a solution to the solar neutrino puzzle. It can be explained by considering the time evolution of a quantum-mechanical system that has two energy eigenstates $|1\rangle$ and $|2\rangle$ with corresponding energy eigenvalues E_1 and E_2. The neutrino flavors correspond to the states $|a\rangle$ and $|b\rangle$ defined as follows:

$$|a\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}} \quad , \quad |b\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}.$$

The system starts in the state $|a\rangle$ at $t = 0$.

(a) Find the wavefunction of the system at $t > 0$.

(b) What is the probability $P_{a\to a}$ to find the system in the state $|a\rangle$ at time t? At which times $P_{a\to a}$ reaches its maximum value?

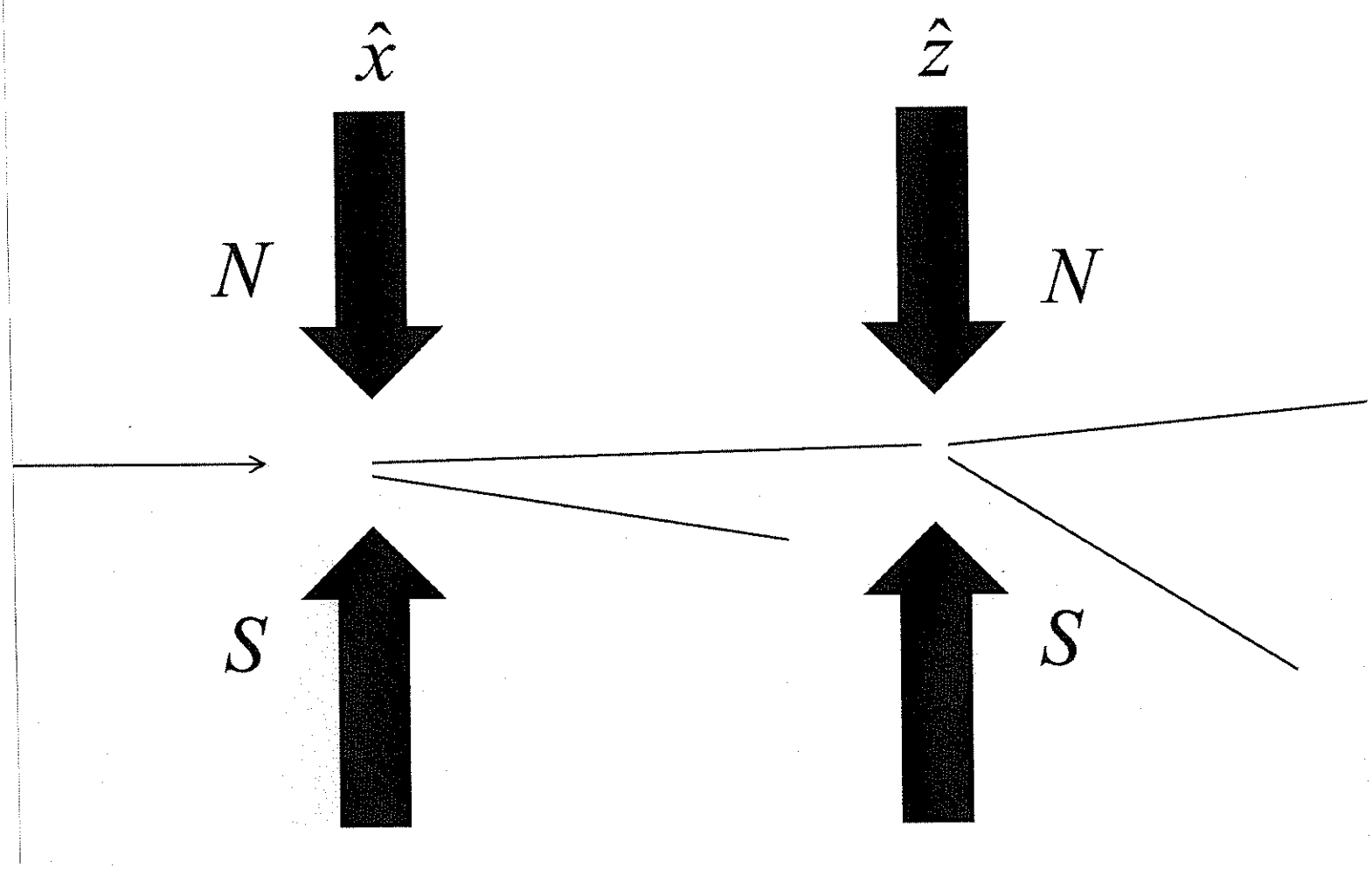
(c) What is the probability $P_{a\to b}$ to find the system in the state $|b\rangle$ at time t? At which times $P_{a\to b}$ reaches it maximum value?

5– An electron is in the spin-up state **along z** at $t = 0$. It then goes through two successive Stern-Gerlach apparatuses the second rotated by 90 degrees from the first as indicated in the figure.

(a) What is the probability for the electron to be found in the spin-up state **along x** after going through the first apparatus?

(b) After the electron is measured along the **x-axis** and found to be spin-up, it goes through the second apparatus. What is the probability for the electron to be found in the spin-down state **along z**?

If N electrons start in the spin-up state **along z** at $t = 0$, for large N what fraction will be found in the spin-down state **along z** after passing through the two Stern-Gerlach apparatuses?



6– Consider a system consisting of an electron and a positron in three-dimensional space of infinite volume including the center-of-mass and relative coordinates. Ignoring the spin of particles:

(a) Write the Hamiltonian for this system.

(b) What are all quantum numbers needed to specify a unique energy eigeenstate?

(c) Find the energy of the ground state.

7– A coherent state represents the closest quantum-mechanical wave-packet to a classical motion. It is constructed from the eigenstates of a harmonic oscillator as follows:

$$|\psi\rangle = \exp(-|c|^2/2) \sum_{n=0}^{\infty} \frac{c^n}{\sqrt{n!}} |n\rangle,$$

where c is an arbitrary complex number, $|0\rangle$ is the ground state and $\mathbf{a}^\dagger$ is the rasing operator. Derive the following properties of the state $|\psi\rangle$:

(a) $|\psi\rangle$ is an eigenstate of the lowering operator $\mathbf{a}$ with the eigenvalue c:

$$\mathbf{a}|\psi\rangle = c|\psi\rangle.$$

(b)

$$\bar{N} \equiv \langle\psi|(\mathbf{a}^\dagger\mathbf{a})|\psi\rangle = |c|^2.$$

(c)

$$(\Delta N)^2 \equiv \langle\psi|(\mathbf{a}^\dagger\mathbf{a})^2|\psi\rangle - \bar{N}^2 = |c|^2.$$

(d)

$$\frac{\Delta N}{\bar{N}} \to 0 \qquad \text{as} \quad \bar{N} \to \infty.$$

8– Two identical non-interacting spin-1/2 particles of mass m are inside a cubic box with hard walls (i.e. infinite potential at the walls). The volume of the box is $V = L^3$. Write the wavefunction for the ground state of this system including the spatial and spin parts.

9– A system consists of two spin-1 particles and its Hamiltonian has no spin dependence.

(a) What is the degeneracy of each energy level of this system due to the spin degree of freedom?

(b) The Hamiltonian now has an additional term $H_S = a\vec{S}_1 \cdot \vec{S}_2$, where $\vec{S}_1$ and $\vec{S}_2$ are the spins of the two particles. Find the energy splitting among degenerate states in part (a) as a result of this interaction.

10– The Hamiltonian for a harmonic oscillator in two dimensions is given by:

$$H_0 = \frac{P_x^2 + P_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2). \tag{1}$$

(a) Find the energy eigenvalues of this oscillator. What is the energy and degeneracy of the first excited level?

(b) Now we add a term $H_1 = Axy$ to the potential ($A \ll m\omega^2$ is a constant). Find the energy eigenvalues and eigenstates of the first level in this case.

Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy
University of New Mexico

Spring 2010

Instructions:

- *the exam consists of 10 problems, 10 points each;*
- *partial credit will be given if merited;*
- *personal notes on two sides of* 8×11 *page are allowed;*
- *total time is 3 hours.*

Table of Constants and Conversion factors

Quantity	Symbol	$\approx$ value
speed of light	c	3.00×10^8 m/s
Planck	h	6.63×10^{-34} J·s
electron charge	e	1.60×10^{-19} C
e mass	m_e	0.511 Mev/$c^2 = 9.11 \times 10^{-31}$ kg
p mass	m_p	938 Mev/$c^2 = 1.67 \times 10^{-27}$ kg
amu	u	931.5 Mev/$c^2 = 1.66 \times 10^{-27}$ kg
fine structure	α	1/137
Boltzmann	k_B	1.38×10^{-23} J/K
Bohr radius	$a_0 = \hbar/(m_e c\alpha)$	0.53×10^{-10} m
conversion constant	$\hbar c$	200 eV·nm = 200 MeV·fm
conversion constant	k_BT@ 300K	1/40 eV
conversion constant	1eV	1.60×10^{-19} J

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1. The earliest conception of the neutron was as a bound state of electron and proton. Suppose some force existed that could confine the electron to the inside of the proton. Estimate the energy of the electron in the ground state.
2. The carbon monoxide molecule absorbs radiation at a wavelength of 2.6 millimeters, corresponding to the rotational excitation of the molecule from $\ell = 0$ to $\ell = 1$. Calculate the molecular bond length.
3. Derive the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$$

for the Schrōdinger equation in 1 dimension. What is the physical significance of ρ?

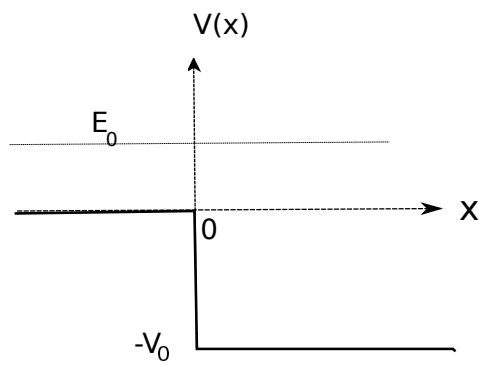
4. A particle is confined to a 1D box $0 < x < a$ and at $t = 0$ has the wave function:

$$\psi(x) = 2\sqrt{\frac{3}{a}}\left(\frac{x}{a}\right), x < \frac{a}{2}$$

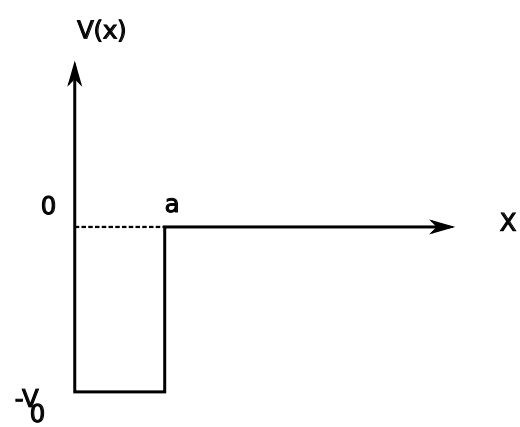
$$\psi(x) = 2\sqrt{\frac{3}{a}}\left(1 - \frac{x}{a}\right), x > \frac{a}{2}$$

Find the expectation value of the energy for this state.

5. An electron with energy $E_0 = V_0/3$ is incident from the left on a step-down potential: $V(x) = 0$ for $x < 0$ and $V(x) = -V_0$ for $x > 0$. What is the transmission probability T?



6. Consider the ground state of a particle in 1 dimension bound in the potential sketched below (v(x) is infinite for $x < 0$, $= -V_0$ for $0 < x < a$ and zero for $x > a$).
(a) Make a sketch of the ground state wave function.
(b) What is the minimum depth V_0 such that at least one bound state exists?



7. Two spin-1/2 particles interact via the Hamiltonian

$$\hat{H} = \frac{2A}{\hbar^2}\left(\hat{S}_x^1\hat{S}_x^2 + \hat{S}_y^1\hat{S}_y^2\right)$$

(a) Explain why the z-component of the *total spin* $\hat{S}_z = \hat{S}_z^1 + \hat{S}_z^2$ is conserved.
(b) In the basis of *total spin* $|s, m\rangle$ (i.e. eigenstates of $\hat{S}^2$, $\hat{S}_z$) the Hamiltonian is diagonal. Find the energy of the ground state.

8. Consider an electron bound in a harmonic oscillator potential with a linear perturbation.

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \epsilon\frac{\hat{x}}{x_0}$$

where $x_0 = \sqrt{\hbar/m\omega}$ and $\epsilon << \hbar\omega$. What is the correction to the ground state energy to second order due to this perturbation? Recall that

$$\hat{x} = \frac{x_0}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right)$$

9. Consider a particle in a harmonic oscillator potential. For an initial state of the particle given by a superposition of energy eigenstates,

$$|\Psi(x,0)\rangle = \frac{1}{\sqrt{2}}(|n\rangle + |n+1\rangle).$$

Calculate $\langle x\rangle$ as a function of time. Recall that the eigenstate energies are $E_n = \hbar\omega(n + 1/2)$ and use the expression for $\hat{x}$ from the previous problem.

10. Consider the spin-orbit coupling of Hydrogen in the n=3 excited state. Label the degenerate multiplets as $3L_j$, where $L = S, P, D$ corresponding to $\ell = 0, 1, 2$, and j is the total angular momentum quantum number. Give the degeneracies of each multiplet.

Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy
University of New Mexico

Spring 2011

Instructions:

•The exam consists of 10 problems (10 points each).
•Where possible, show all work; partial credit will be given if merited.
•Personal notes on two sides of an 8×11 page are allowed.
•Total time: 3 hours.

Useful Information:

1. Pauli sigma matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2. $\int_0^{\pi/2} dx \sin(2x)\sin(x) = -2/3$

3. For a particle with mass m moving under the influence of an attractive one-dimensional square-well potential, $V(x) = -V_0$ for $|x| \leq a$ and $V(x) = 0$ otherwise, the energy eigenvalues for the bound states are determined from the transcendental equation

$$\frac{\sqrt{\lambda - y^2}}{y} = \begin{cases} \tan y & \text{(even parity)} \\ -\cot y & \text{(odd parity)} \end{cases},$$

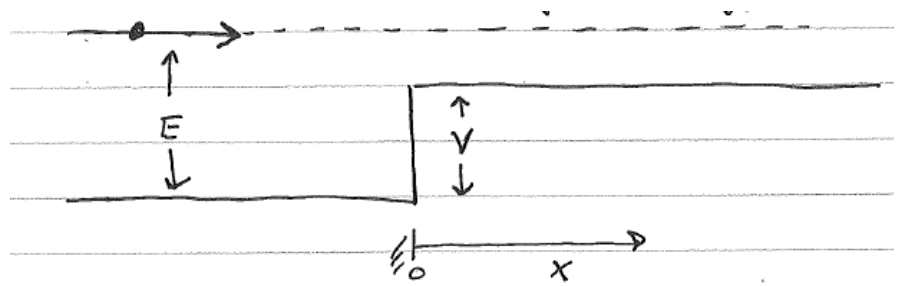
where $\lambda = 2ma^2V_0/\hbar^2$ and $y = qa$ with $q = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$.

4. Some hydrogen wave functions:

Table 1: Wave functions and their components

n	ℓ	m	$R_{n\ell}$	$Y_{\ell m}$	$\psi_{n\ell m} = R_{n\ell} Y_{\ell m}$
1	0	0	$2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$
2	0	0	$\left(\frac{1}{2a_0}\right)^{3/2}\left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2}\left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$
2	1	0	$\left(\frac{1}{2a_0}\right)^{3/2} \frac{1}{\sqrt{3}}\frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$	$\frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2} \frac{r}{a_0} e^{-r/2a_0}\cos\theta$
2	1	± 1	$\left(\frac{1}{2a_0}\right)^{3/2} \frac{1}{\sqrt{3}}\frac{r}{a_0} e^{-r/2a_0}$	$\pm\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{\pm i\phi}$	$\frac{1}{8}\sqrt{\frac{1}{\pi}}\left(\frac{1}{a_0}\right)^{3/2} \frac{r}{a_0} e^{-r/2a_0}\sin\theta e^{\pm i\phi}$

P1. A particle with mass m moving to the right with kinetic energy E encounters a step potential of height V, as shown in the figure. What is the probability that the particle will be reflected if $E = (4/3)\,V$?



P2. Consider a beam of electrons, to be fired a great distance, $L = 10^4$ km, along the x axis. In momentum-space the electron wavefunction is a strongly peaked Gaussian at $\hbar k_0$ with energy $\hbar\omega = \frac{(\hbar k_0)^2}{2m} = 10$ eV. The real-space wavefunction evolves in time according to

$$\psi(x,t) = A \exp\left[i\left(k_0 x - \omega t\right)\right] \exp\left[-\frac{(x-\omega' t)^2}{4\left(\sigma^2 + \frac{i}{2}\omega'' t\right)}\right],$$

where A and σ are constants and $\omega' = (\partial\omega/\partial k)|_{k=k_0}$ denotes differentiation of ω with respect to k .

If the initial uncertainty in the position of an electron is $\Delta x = 1.0$ mm, approximately what will be Δx on arrival?

P3. The Hamiltonian for an anharmonic oscillator in one dimension is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + g\hat{x}^4$$

where g is positive. Use the uncertainty principle to estimate the ground state energy. Express your answer as a function of g, $\hbar$, and the mass m.

P4. A particle with mass m is initially in the ground state of a one-dimensional infinite square well with width a. If the square well is suddenly stretched to a new width $2a$, what is the probability that a measurement of the energy will find the particle to be in the ground state of the new square well?

P5. The motion of a particle with mass m moving in one dimension is described by the Hamiltonian

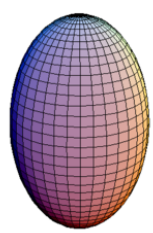
$$\hat{H} = \frac{\hat{p}^2}{2m} + mg\hat{x}.$$

Obtain an expression for the expectation value of the position $\langle \hat{x}(t) \rangle$ in terms of the initial expectation values $\langle \hat{p}(0) \rangle$ and $\langle \hat{x}(0) \rangle$.

P6. The nucleus of a gold atom is found to be aspherical, for its rotational motion is described by the Hamiltonian for the axially symmetric rotator,

$$\hat{H} = \frac{\hat{L}_x^2 + \hat{L}_y^2}{2I_1} + \frac{\hat{L}_z^2}{2I_2}$$

where the moment of inertia $I_1 > I_2$. Sketch the splittings in the rotational energy spectrum for $\ell = 0, 1,$ and 2.



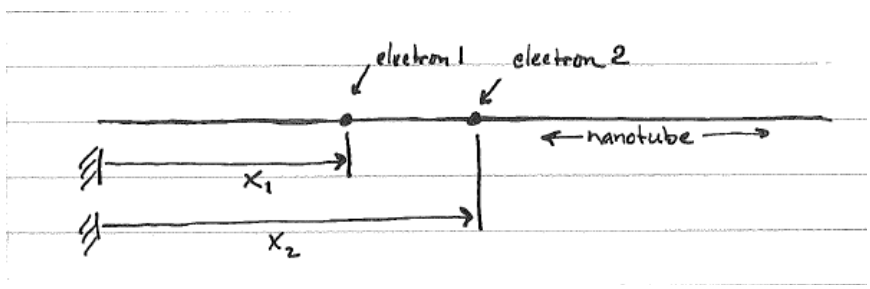
P7. The Hamiltonian for a harmonic oscillator is given by

$$\hat{H} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right),$$

where the operators $\hat{a}$ and $\hat{a}^\dagger$ satisfy the commutation relation $\left[\hat{a}, \hat{a}^\dagger\right] = 1$.

Consider an eigenstate $|n\rangle$ of $\hat{H}$ with eigenvalue $\left(n + \frac{1}{2}\right)\hbar\omega_0$. Prove that the two states $\left|\phi_{(-)}\right\rangle = C_{(-)}\hat{a}\,|n\rangle$ and $\left|\phi_{(+)}\right\rangle = C_{(+)}\hat{a}^\dagger\,|n\rangle$ are also eigenstates of $\hat{H}$, where $C_{(-)}$ and $C_{(+)}$ are normalization constants. Find their respective eigenvalues and normalization constants.

P8. In quasi one-dimensional conductors, the Coulomb repulsion between conduction electrons is screened out at large distances. All that remains is small phonon-mediated interaction. This interaction is attractive, and is often modeled by a square-well potential.



Consider two electrons in a carbon nanotube, each in the same spin state, their center of mass free to move in the x direction. Suppose that they are attracted to one another by the square-well interaction $V(x_1 - x_2) = -V_0$ for $|x_1 - x_2| \leq a$ and $V(x_1 - x_2) = 0$ otherwise, where the argument $x_1 - x_2$ is the difference between the displacements x_1 and x_2 of electron 1 and electron 2 respectively. What is the minimum depth V_0 required for a bound state for the two electron system? You may want to make use of the square-well solution listed in the table.

P9. An electron initially in an eigenstate of $\hat{S}_x$ with eigenvalue $\hbar/2$ is placed in a uniform magnetic field $\vec{B} = B_0\ \hat{z}$. At some later time t a measurement is to be made of $\hat{S}_y$. What is the probability that a value $\hbar/2$ will be found?

P10. If the photon actually had a small mass m^*, electrons and protons would attract one another via a screened (Yukawa) potential of the form

$$U(r) = \frac{-e^2}{4\pi\epsilon_0 r} e^{-\gamma r},$$

where the screening parameter $\gamma = m^* c/\hbar$. This would show up as causing a shift in the hydrogen spectrum.

To first order in γ, calculate the shift in the energy $E_{0,0,0}$ of the hydrogen ground state. What is the first order shift in the energy $E_{n,\ell,m}$ for arbitrary quantum numbers n, ℓ, and m?

Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy
University of New Mexico

Spring 2012

Instructions:

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•Personal notes on two sides of an 8×11 page are allowed.
•Total time: 3 hours.

Useful Information:

1. Pauli sigma matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
2. Some hydrogen wave functions:

Table 1: Wave functions and their components

n	ℓ	m	$R_{n\ell}$	$Y_{\ell m}$	$\psi_{n\ell m} = R_{n\ell}\, Y_{\ell m}$
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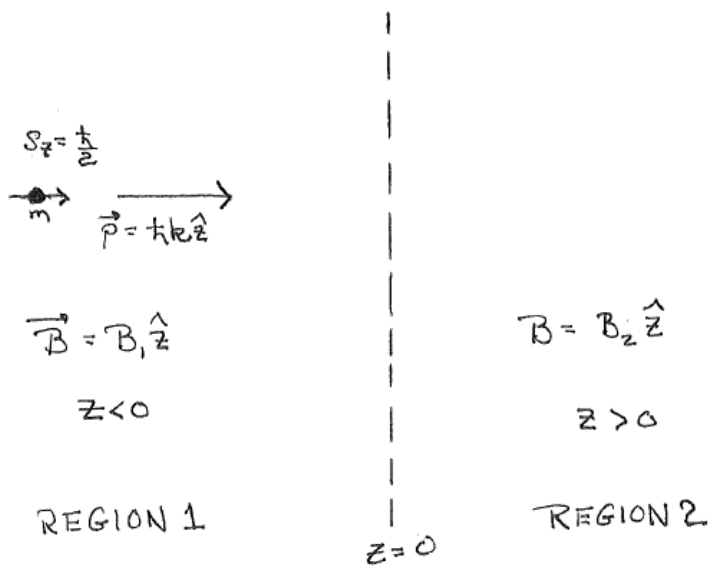
2	0	0	$\left(\frac{1}{2a_0}\right)^{3/2}\left(2-\frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2}\left(2-\frac{r}{a_0}\right) e^{-r/2a_0}$
2	1	0	$\left(\frac{1}{2a_0}\right)^{3/2} \frac{1}{\sqrt{3}}\frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$	$\frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2} \frac{r}{a_0} e^{-r/2a_0}\cos\theta$
2	1	± 1	$\left(\frac{1}{2a_0}\right)^{3/2} \frac{1}{\sqrt{3}}\frac{r}{a_0} e^{-r/2a_0}$	$\pm\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{\pm i\phi}$	$\frac{1}{8}\sqrt{\frac{1}{\pi}}\left(\frac{1}{a_0}\right)^{3/2} \frac{r}{a_0} e^{-r/2a_0}\sin\theta e^{\pm i\phi}$

3. Hydrogen expectation values: $\left\langle \frac{1}{r^3} \right\rangle_{n,\ell} = \frac{1}{a_0^3 n^3}\frac{1}{\ell(\ell+1/2)(\ell+1)}$
4. Commutation relations: $[A, BC] = B\,[A, C] + [A, B]\,C$
5. Lorentzian Fourier Transform: $\int_{-\infty}^{\infty} dx\, \exp(i\omega x)\frac{\alpha/\pi}{\alpha^2+x^2} = \exp(-|\alpha\omega|)$

P1. An electron with mass m and spin $S_z = \frac{\hbar}{2}$ moves in the z direction, parallel to a magnetic field $\vec{B} = B\hat{z}$. For $z < 0$, the magnetic field is uniform, with $B = B_1$, and for $z > 0$ it is again uniform, with $B = B_2$. The Hamiltonian,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \begin{cases} \frac{2\mu B_1}{\hbar}\hat{S}_z; & z < 0 \\ \frac{2\mu B_2}{\hbar}\hat{S}_z; & z > 0 \end{cases},$$

is time-independent, and effectively discontinous at $z = 0$, assuming that the distance over which B changes from B_1 to B_2 is sufficiently short. The electron is initially located to the left of the origin, and moves to the right, in the z direction, with momentum $\hbar k$. If its spin does not flip, what is the probability P that the electron will cross the origin and continue moving to the right? For what conditions will $P = 0$?



P2. The spin-orbit coupling for the hydrogen atom is described by the interaction Hamiltonian

$$H_{\rm so} = \left(1.77 \times 10^{-10}\text{eV}\right) \left(\frac{a_0^3}{r^3}\right) \frac{\vec{S} \cdot \vec{L}}{\hbar^2}$$

To first order, how does the spin-orbit interaction perturb the energy levels of the the $2p$ state? Express your answers in eV. The Bohr radius is denoted by a_0.

P3. Consider a particle with mass m confined to a linear potential $V(r) = kr$ in three dimensions. For $\ell = 0$, the radial Schroedinger equation is

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + kr\right) u(r) = Eu(r)$$

where $u(r) = r\psi(r)$ satisfies boundary conditions $u(0) = u(\infty) = 0$. Using a transformation to dimensionless variables, show that the allowed energies must be proportional to

$$\varepsilon = \left(\frac{\hbar^2 k^2}{2m}\right)^{1/3}.$$

P4. A tritium atom consists of one electron bound to a triton [$H^3 = (nnp)$]. The triton is unstable and decays into a He^3 nucleus by converting a neutron to a proton while casting off an electron and an antineutrino [(nnp)$\rightarrow$ (npp)+$e^- + \bar{\nu}_e$]. As compared to the period of the orbiting electron, this nuclear decay occurs practically instantaneously ($t < 10^{-13}$ sec). If the original electron remains bound to the nucleus, a positively charged He^3 ion is formed. What is the probability that the ion will be in its ground state? It might help to you to recall the ground state hydrogen wave function, and the integral

$$\int_0^\infty \exp(-\alpha x) x^n dx = \frac{n!}{a^{n+1}}.$$

P5. The expectation value $\langle \vec{r} \cdot \vec{p} \rangle$ may be shown to be a constant for any stationary state. From the equation of motion, $\frac{d}{dt} \langle \vec{r} \cdot \vec{p} \rangle = \frac{i}{\hbar} \langle [H, \vec{r} \cdot \vec{p}] \rangle$, show that

$$\left\langle \frac{p^2}{m} \right\rangle = \left\langle \vec{r} \cdot \vec{\nabla} V(\vec{r}) \right\rangle$$

for a particle moving in a potential $V(\vec{r})$. For the ground state of hydrogen, $\langle H \rangle = \left\langle \frac{p^2}{2m} - \frac{e^2}{r} \right\rangle = -13.6$ eV. Use this, together with the relation above, to determine the value of $\left\langle \frac{p^2}{2m} \right\rangle$.

P6. Consider a particle confined by an infinite square well $0 < x < a$ in one dimension. The initial wave function

$$\psi(x,0) = \frac{1}{\sqrt{a}}\left(\sin\frac{\pi x}{a} + i\sin\frac{2\pi x}{a}\right)$$

is a complex superposition of the ground state and the first excited state. Calculate the probability flux. In which direction will the particle most likely be moving at $x = a/2$, to the right, or to the left?

P7. The Hamiltonian for a harmonic oscillator is given by

$$\hat{H} = \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right).$$

Recall that the operators $\hat{a}$ and $\hat{a}^\dagger$ satisfy the commutation relation $\left[\hat{a},\hat{a}^\dagger\right] = 1$, that the position operator $\hat{x} = \left(\hat{a}^\dagger + \hat{a}\right)\sqrt{\frac{\hbar}{2m\omega_0}}$, and that the momentum operator $\hat{p} = i\left(\hat{a}^\dagger - a\right)\sqrt{\frac{\hbar m\omega_0}{2}}$. Suppose that the oscillator is prepared in an initial state

$$|\psi(0)\rangle = \exp\left(g\hat{a}^\dagger\right)|0\rangle\exp\left(-g^2/2\right),$$

where g is real. What is the physical significance of this state? Calculate $\langle\hat{x}(t)\rangle$.

P8. Consider two noninteracting electrons moving in a one dimensional infinite square well of width a in a uniform magnetic field $\vec{B} = B\hat{z}$. The Hamiltonian describing this system is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{2\mu}{\hbar}\left(\vec{S}_1 + \vec{S}_2\right) \cdot \vec{B}$$

for $0 < x < a$. The subscripts 1 and 2 refer to the two electrons, respectively. The particle-in-a-box eigenfunctions are $\psi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$. Write down an expression for the two-particle ground state wavefunction when $B = 0$. Be sure to include a description of both space and spin. Write down an expression for the first excited state, and describe any degeneracies. Which of these becomes the new ground state at sufficiently high B?

P9. A particle with spin $\frac{1}{2}$ is at rest in a magnetic field

$$\vec{B} = B_0\left(\hat{x}\cos\phi + \hat{y}\sin\phi\right)$$

Its Hamiltonian is given by

$$H = -\gamma\vec{B} \cdot \vec{S}$$

Here $\hat{x}$ and $\hat{y}$ are unit vectors, and $\vec{S}$ is the spin operator. At time t=0 the particle is in state $\left|m_s = \frac{1}{2}\right\rangle$, where $m_s\hbar$ is the eigenvalue of the operator S_z. Calculate the probability that, at some later time t, the system will be found in the state $\left|m_s = -\frac{1}{2}\right\rangle$.

P10. Under the action of a time-dependent Hamiltonian, $H = H_0 + H_1(t)$, to first order in H_1, the state $|\psi(t_2)\rangle$ is connected to the state $|\psi(t_1)\rangle$ through the integral expression,

$$|\phi(t_2)\rangle \simeq |\phi(t_1)\rangle + \frac{1}{i\hbar}\int_{t_1}^{t_2} dt\; e^{iH_0 t/\hbar} H_1(t) e^{-iH_0 t/\hbar} |\phi(t_1)\rangle \,,$$

where $|\phi(t)\rangle = e^{iH_0 t/\hbar} |\psi(t)\rangle$ is the state vector in the interaction representation.

Consider a two-state system for which

$$H_0 = \varepsilon_1 |1\rangle \langle 1| + \varepsilon_2 |2\rangle \langle 2| \,.$$

The system is subjected to a perturbation

$$H_1(t) = \frac{V_0}{1+\gamma^2 t^2} \left(|1\rangle \langle 2| + |2\rangle \langle 1| \right)$$

that turns on and off with a rate γ as t goes from $-\infty$ to ∞. If $|\psi(-\infty)\rangle = |1\rangle$, to first order in V_0, what is the probability that $|\psi(\infty)\rangle = |2\rangle$? Show that the adiabatic theorem is satisfied when $H_1(t)$ is slowly varying.

Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy
University of New Mexico

Spring 2013

Instructions:

•The exam consists of 10 problems (10 points each).
•Where possible, show all work; partial credit will be given if merited.
•Personal notes on two sides of an 8×11 page are allowed.
•Total time: 3 hours.

Useful Formulae and Constants:

$$\int_{-\infty}^{\infty} dx \exp(-ax^2)x^{2n} = \frac{(2n+1)!!\sqrt{\pi}}{2^n a^{\frac{2n+1}{2}}}$$

$$\begin{aligned}
\psi_{0,0,0}(r,\theta,\phi) &= \frac{1}{\sqrt{\pi a_0^3}} \exp\left(-\frac{r}{a_0}\right); \\
a_0 &= \frac{4\pi\epsilon_0\hbar^2}{me^2}
\end{aligned}$$

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z$$

$$\begin{aligned}
\hat{x} &= \sqrt{\frac{\hbar}{2m\omega_0}}\left(b + b^\dagger\right) \\
\hat{p} &= -i\sqrt{\frac{\hbar m\omega_0}{2}}\left(b - b^\dagger\right)
\end{aligned}$$

$$\begin{aligned}
\psi_n(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right); \\
E_n &= \frac{n^2\pi^2\hbar^2}{2mL^2}
\end{aligned}$$

P1. Consider the time-independent Schroedinger equation for a one dimensional forced harmonic oscillator,

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \left(\frac{1}{2}m\omega^2 x^2 + Fx\right)\psi = E\psi.$$

By making an appropriate coordinate transformation, determine how the energy eigenvalues depend on the constant force F.

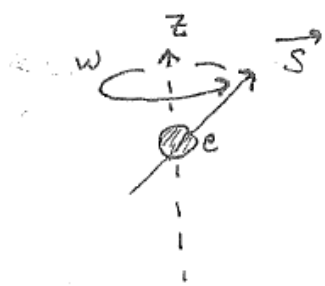
P2. Consider the Hamiltonian for the motion of the spin of an electron in a constant magnetic field B,

$$\hat{H} = -2\frac{\mu}{\hbar}B\ \hat{S}_z.$$

By solving Heisenberg's equations of motion, show that the x component of the spin operator evolves in time, such that

$$\hat{S}_x(t) = \hat{S}_x \cos(\omega t) + \hat{S}_y \sin(\omega t)$$

where $\omega = 2\frac{\mu}{\hbar}B$.



P3. Consider a two-level atom, with ground state $|1\rangle$, and first excited state $|2\rangle$. In the presence of an electric field E_0, the Hamiltonian is given by

$$\hat{H} = \frac{\Delta}{2}\left(|2\rangle\langle 2| - |1\rangle\langle 1|\right) + p_0 E_0\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right)$$

where p_0 is the transition-dipole matrix element. What are the eigenstates and eigenvalues of $\hat{H}$?

P4. The time-independent Schroedinger equation for the hydrogen atom, written explicitly in spherical coordinates, is as follows:

$$\frac{-\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi-\frac{e^2}{4\pi\epsilon_0 r}\psi=E\psi$$

Apply the method of separation of variables to reduce this *partial* differential equation to three *ordinary* differential equations. Assume that the wave function $\psi(r,\theta,\phi)=R(r)P(\theta)F(\phi)$ can be written in a product form, introduce appropriate separation constants, and obtain (but do not solve) three equations, one for each of the functions $R(r)$, $P(\theta)$, and $F(\phi)$, in their respective variables r,θ, and ϕ.

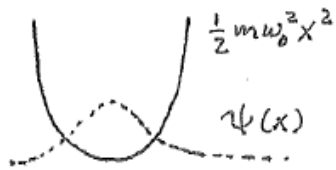
P5. Consider a particle with mass m confined to a harmonic oscillator potential in one dimension,

$$U(x) = \frac{1}{2} m\omega_0^2 x^2.$$

Use the variational method to find a lower bound for the ground state energy E_0. Take the trial wavefunction to be $\psi(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp(-\frac{\alpha}{2}x^2)$, with α to be determined, and show that the bound,

$$E_0 \geq \frac{\hbar\omega_0}{2},$$

you obtain variationally agrees with the exact ground state energy. What does this imply for the trial wavefunction?



P6. Consider the Coulomb potential

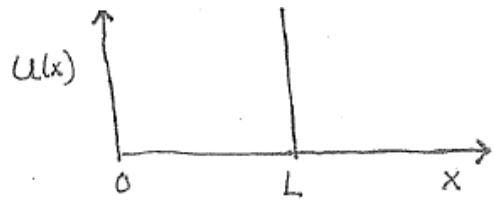
$$U_0(r) = -\frac{e^2}{4\pi\epsilon_0 r}$$

describing the attraction between the electron and the proton in the hydrogen atom. Due to the finite radius $a \simeq 2 \times 10^{-15}$ m of the proton, one expects modifications to this potential at short range. In one proposal, the proton is to be modeled as a point charge surrounded by a spherical "skin" consisting of a double-layer at the radius a. The effect of the double-layer is to introduce a discontinuity Δ, such that the actual potential is given by

$$U(r) = \begin{cases} U_0(r); r > a \\ U_0(r) + \Delta; r \leq a \end{cases}$$

To first order, what is the shift in the ground state energy for $\Delta \simeq 1$ eV ? Note that $\exp\left(-\frac{r}{a_0}\right) \simeq 1$ for $r << a_0$.

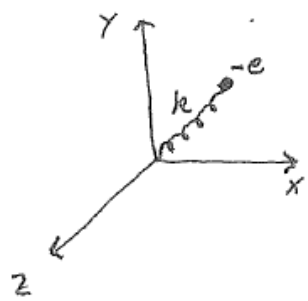
P7. Consider a particle of mass m confined to a one-dimensional box with width L. The particle is initially in the ground state. The box is *suddenly* doubled in size, from L to $2L$. Show that the probability to find the particle in the first excited state of the new box is $1/2$.



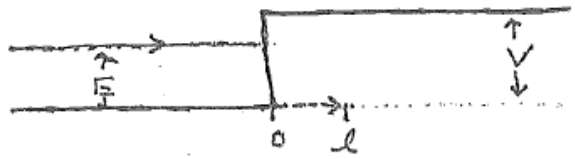
P8. An atom subjected to an electric field E will acquire an induced dipole moment $p = \alpha E$, where α is the atomic polarizability, thereby lowering its electronic energy by

$$\Delta\varepsilon = -\frac{1}{2}\alpha E^2.$$

Consider a model of an atom consisting single electron with mass m and charge e, confined by a three-dimensional harmonic oscillator potential with frequency ω_0. Assume that the atom is subjected to an electric field in the x-direction, so that the total potential energy is $U(\vec{r}) = \frac{1}{2}m\omega_0^2 r^2 + eEx$. By considering the perturbation of the ground state, find an expression for the atomic polarizability α.



P9. A free particle with mass m traveling to the right with energy E is reflected from a square barrier of height V, as shown in the figure. What is the probability that the particle will penetrate the barrier to a depth greater than a given value ℓ?



P10. Two noninteracting electrons are confined to a square well potential $U(x)$ in one dimension. For $x > L$, or $x < 0$, the potential is infinite. For $0 < x < L$, the potential is independent of position, but dependent on spin, such that

$$U(x) = -\frac{\pi^2\hbar^2}{mL^2}\left(2\,\frac{\vec{S}_1 \cdot \vec{S}_2}{\hbar^2} + \frac{1}{2}\right)$$

Find the ground state wavefunction and its associated energy for the two-particle system (include both space and spin)?

Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy

University of New Mexico

Spring 2014

Instructions:

- The exam consists of 10 short-answer problems (10 points each).
- Where possible, show all work; partial credit will be given if merited.
- Personal notes on two sides of an 8×11 page are allowed. For some problems it may be helpful to use the formulae provided below.

Useful Formulae:

$$E = \left(n + \frac{1}{2}\right)\hbar\omega \ ; \ \hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}}\left(b + b^\dagger\right) \ ; \ \hat{p} = -i\sqrt{\frac{\hbar m\omega_0}{2}}\left(b - b^\dagger\right)$$

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{1}{\pi x_0^2}\right)^{1/4} e^{-\frac{x^2}{2x_0^2}} H_n\left(x / x_0\right); \ x_0 = \sqrt{\frac{\hbar}{m\omega}} \ ; \ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

P1. A particle with mass m is incident from the left $(x < 0)$ on a step potential,

$$V(x) = V_0, \quad x > 0$$
$$V(x) = 0, \quad x < 0$$

If the particle has energy $E < V_0$, the wave function for $x < 0$ can be written in the form

$$\psi(x) = e^{ikx} - e^{i\delta}e^{-ikx} \ .$$

Derive an expression for the phase δ in terms of $E, m,$ and V_0.

P2. Consider a particle in a one-dimensional box $-a/2 < x < a/2$ in the state

$$\psi(x) = \sqrt{\frac{5}{a^5}}\left(x - \frac{a}{2}\right)^2, \quad -a/2 < x < a/2$$

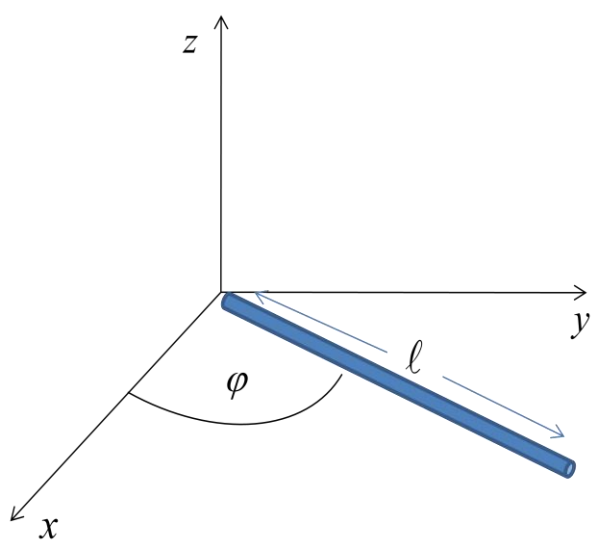
Evaluate the expectation value of the energy of the particle in this state, and show that this is greater than the ground state energy.

P3. Consider a two-state system for which the Hamiltonian is

$$\hat{H} = iA\left(|1\rangle\langle 2| - |2\rangle\langle 1|\right) ,$$

where A is real. At time $t = 0$ the system is in the state $|1\rangle$. What is the probability that the system will be in the same state at a later time t?

P4. A thin rod with mass m and length ℓ is constrained to lie in the xy plane with one end fixed at the origin. It is free to rotate through an angle φ about the z-axis, as shown in the figure. What is the Hamiltonian for this system? If the rod is prepared in a state of definite angular momentum $\hbar\hat{z}$, what is its wave function $\psi(\varphi)$? What is the expectation value $\langle\dot{\varphi}\rangle$ of its angular velocity? [The moment of inertia is $(1/3)m\ell^2$].



P5. Consider a particle with mass m moving in one dimension, its dynamics described by the harmonic oscillator Hamiltonian, $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2$. For an initial state given by a superposition of energy eigenstates, $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|n\rangle + |n+1\rangle\right)$, determine the time evolution of the expectation value $\left\langle \frac{1}{2}k\hat{x}^2 \right\rangle$ of the potential energy.

P6. A particle moves in a three-dimensional spherical "square-well" potential,

$$V(r) = \begin{cases} 0, & r \le R \\ \infty, & \text{r>R} \end{cases}$$

Its dynamics is determined by the Hamiltonian $\hat{H} = \frac{1}{2m}\left(\hat{P}_r^{\,2} + \frac{\hat{L}^2}{r^2}\right) + V(r)$, where $\hat{P}_r = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r}r$

Taking the wave function to be of the form, $\psi(\vec{r}) = \frac{u(r)}{r}Y_{\ell,m}(\varphi,\theta)$ in spherical coordinates, show that the radial Schroedinger equation is

$$\frac{-\hbar^2}{2m}u''(r) + V(r)u(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}u(r) = Eu(r) \ .$$

Explain, either mathematically, or intuitively, why ℓ should be zero in the ground state. Write down an expression for $u(r)$ in the ground state, and determine its associated energy eigenvalue.

P7. A photon is in the state $|\psi\rangle = \frac{1}{\sqrt{5}}\left(|x\rangle - 2i|y\rangle\right)$ where $|x\rangle$ and $|y\rangle$ refer to photon states polarized along the x and y axes, respectively. What is the probability for the photon to be right circularly polarized?

P8. Two electrons are confined to a harmonic oscillator potential in the x direction and are subjected to a uniform magnetic field $\vec{B} = B_0\hat{z}$. Ignore electron-electron interactions and ignore orbital magnetic coupling. The Hamiltonian is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{2\mu_B}{\hbar}\left(\vec{S}_1 + \vec{S}_2\right)\cdot\vec{B},$$

where the subscripts 1 and 2 refer to particles 1 and 2, respectively, and μ_B is the Bohr magneton. By appropriately combining harmonic oscillator eigenfunctions, write down explicit expressions for the two-body wave functions and their associated energies for the ground state and the first excited state(s). Include both space and spin degrees of freedom.

P9. A particle moving in two-dimensions is confined by an infinite square-well potential to the region $-L/2 < x < L/2,\ -L/2 < y < L/2$. The eigenfunctions for the time-independent Schroedinger equation may be written in four different ways;

$$\psi_{m,n}(x,y) = \frac{2}{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{L}\right);$$
$$\psi_{m,n}(x,y) = \frac{2}{L}\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right);$$
$$\psi_{m,n}(x,y) = \frac{2}{L}\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{L}\right);$$
$$\psi_{m,n}(x,y) = \frac{2}{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right);$$

the particular choice depends on whether *m* and *n* are even or odd. What is the energy and degeneracy of the first excited state, and what are the eigenfunctions? If the square-well is perturbed with a spatially-dependent potential $V(x,\mathrm{y}) = \varepsilon_0 \frac{xy}{L^2}$, where $\varepsilon_0 << \hbar^2\pi^2/2mL^2$, what is the splitting of the first excited state, to lowest (non-vanishing) order?

[Useful Integral: $\int_{-\pi}^{\pi} \sin(\varphi)\cos(\varphi/2)\varphi d\varphi = \frac{32}{9}$]

P10. An electron is prepared in the spin-up state with respect to the z axis. Consider a second axis z' that is titled from the z axis by an angle of 30^0. What is the probability that a measurement of the spin along the z' axis will give a value of $\hbar/2$?

Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy

University of New Mexico

Spring 2015

Instructions:

- The exam consists of 10 problems (10 points each).
- Where possible, show all work; partial credit will be given if merited.
- Useful formulae are provided below; crib sheets are not allowed.

Useful Formulae:

$$E = \left(n + \frac{1}{2}\right)\hbar\omega \;;\; \hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}}\left(b + b^\dagger\right) \;;\; \hat{p} = -i\sqrt{\frac{\hbar m\omega_0}{2}}\left(b - b^\dagger\right)$$

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{1}{\pi x_0^2}\right)^{1/4} e^{-\frac{x^2}{2x_0^2}} H_n\left(x / x_0\right);\; x_0 = \sqrt{\frac{\hbar}{m\omega}} \;;\; H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z \;,\; D(\hat{n},\theta) = \exp\left(-i\frac{\theta}{\hbar}\hat{n}\cdot\vec{S}\right)$$

$$\psi_{100} = \frac{1}{\sqrt{4\pi}} \quad \times \quad \frac{2}{a_0\sqrt{a_0}} e^{-r/a_0}$$

$$\psi_{200} = \frac{1}{\sqrt{4\pi}} \quad \times \quad \frac{1}{2\sqrt{2}a_0\sqrt{a_0}}(2 - \frac{r}{a_0})e^{-r/2a_0}$$

$$\psi_{210} = \sqrt{\frac{3}{4\pi}}\cos\theta \quad \times \quad \frac{1}{2\sqrt{6}a_0\sqrt{a_0}}(\frac{r}{a_0})e^{-r/2a_0}$$

$$\psi_{21\pm1} = \pm\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}\frac{1}{2\sqrt{6}a_0\sqrt{a_0}}(\frac{r}{a_0})e^{-r/2a_0}$$

$$\psi_{300} = \frac{1}{\sqrt{4\pi}} \quad \times \quad \frac{2}{81\sqrt{3}a_0\sqrt{a_0}}(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2})e^{-r/3a_0}$$

$$\vdots$$

$$1\text{eV} = 1.6\times10^{-19}\,\text{J}$$

$$\frac{e^2}{\hbar c} = \frac{1}{137};\; \hbar c = 197\text{MeV}\cdot\text{fm}$$

$$\int_{-\infty}^{\infty} dx\, x^2\exp(-ax^2) = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

$$\int_{-\infty}^{\infty} dx\exp\left(-ax^2\right)\exp\left(ikx\right) = \sqrt{\pi/a}\exp\left(-\frac{ak^2}{4}\right)$$

$$\int_{-\infty}^{\infty} dx\left(x^2 + a^2\right)^{-1}\exp(ikx) = \frac{\pi}{a}\exp\left(-a|k|\right)$$

P1. A system consists of two spins, $\vec{s}_1$ and $\vec{s}_2$ in a uniform magnetic field $\vec{B} = B_0\hat{z}$ directed along the z axis. Their evolution is governed by the Hamiltonian,

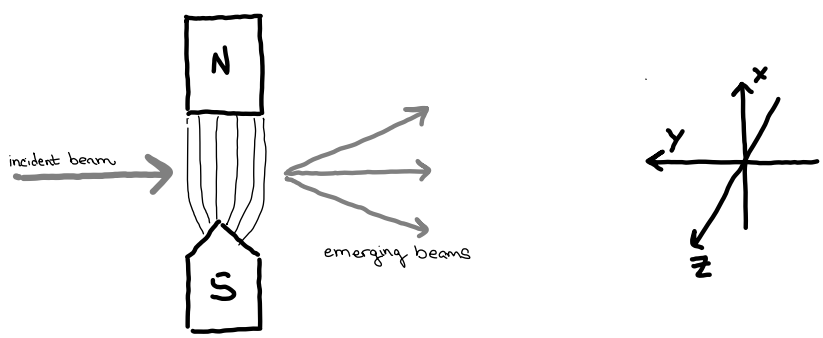
$$H = \frac{\varepsilon}{\hbar^2}\left(\vec{s}_1 \cdot \vec{s}_2\right) - \frac{\mu}{\hbar}\vec{B}\cdot\left(\vec{s}_1 + \vec{s}_2\right)$$

Which of the following quantities are constants of the motion: $\vec{s}_1, \vec{s}_2, \vec{s}_1^{\,2}, \vec{s}_2^{\,2}, \vec{s} = \vec{s}_1 + \vec{s}_2, \vec{s}^{\,2}, s_z$? If both are spin-1/2, what are the eigenvectors and eigenvalues of *H*?

P2. Consider a spin-1 particle. The matrix representation of the components of spin-1 angular momentum are,

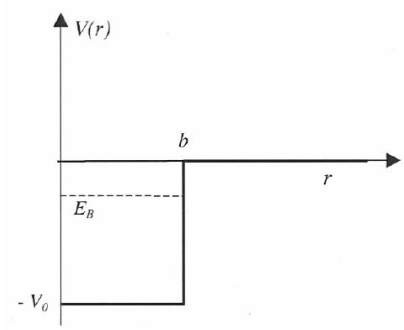
$$\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \hat{S}_y = \frac{\hbar}{i\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \hat{S}_z = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The matrices are expressed in the ordered basis $\left(|+1\rangle, |0\rangle, |-1\rangle\right)$ where the kets are labeled by the eigenvalues of $\hat{S}_z$ in units of $\hbar$.



Show that $|\psi_A\rangle = \frac{1}{2}\left(|+1\rangle + \sqrt{2}|0\rangle + |-1\rangle\right)$, $|\psi_B\rangle = \frac{1}{2}\left(|+1\rangle - \sqrt{2}|0\rangle + |-1\rangle\right)$, and $|\psi_C\rangle = \frac{1}{\sqrt{2}}\left(|+1\rangle - |-1\rangle\right)$ are eigenvectors of $\hat{S}_x$ and find their eigenvalues. When a beam of spin-1 particles initially prepared in the state $|+1\rangle$ is put through a Stern-Gerlach apparatus having the magnetic field gradient in the x-direction, three beams emerge. What are the relative intensities of the three beams?

P3. A deuteron is a bound state of a proton and a neutron, $m_N \simeq m_P \simeq 939$ MeV/c^2. The attraction between the two particles can be modeled by a square-well potential of width b and depth $-V_0$, as illustrated in the figure.



The reduced radial wavefunction $u(r) = rR(r)$ for the ground state is determined from the time-independent Schroedinger equation for motion relative to the center of mass,

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} + V(r)u = Eu$$

Assume a solution of the form $u(r) = A\sin(kr)$ for the region $0<r<b$, and $u(r) = B\exp(-qr)$ for $b \le r$, where A, B are normalization constants; find the relations for k and q in terms of E, and show that E is determined by the transcendental equation $\tan(kb) = -k/q$. If $V_0 = 38.5$ MeV, what is the minimum width b (in fm) below which there are no bound states?

(Note: $\hbar c = 197$ MeV-fm)

P4. The wavefunction for a particle moving freely on the x-axis is given by

$$\psi(x) = \frac{1}{\left(2\pi\sigma^2\right)^{1/4}}\exp\left(-\frac{x^2}{4\sigma^2}\right)\exp\left(ik_0x\right)$$

where σ and k_0 are constants. Calculate the distribution of expected outcomes of a measurement of the particle's energy E. What is the mean of this distribution?

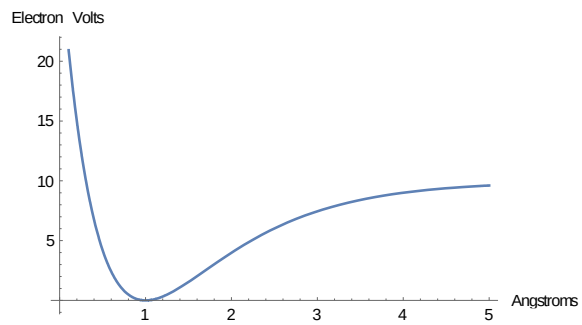
P5. Consider a particle moving in three dimensions. Show which of the following observables can be measured simultaneously:

$$L_x = yp_z - zp_y, L_y = zp_x - xp_z, L_z = xp_y - yp_x, z, p_z, r^2 = x^2 + y^2 + z^2$$

Here (x, y, z) and (p_x, p_y, p_z) are the three Cartesian components of position and momentum respectively, and r is the radius from the origin.

P6. A simple three-parameter potential that is often used to characterize the atomic separation distance r in a diatomic molecule is Morse potential, given by

$$U(r) = D\left(1 - \exp\left(-a\left(r - r_0\right)\right)\right)^2 \ ,$$



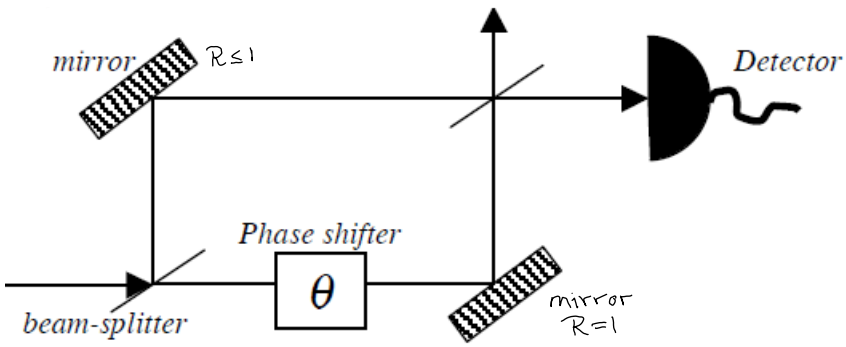
and sketched in the graph. Here r_0 is the equilibrium bond distance, D is the dissociation energy, and *a* is a parameter that determines with width of the potential. For the N_2 molecule in the ground state, the equilibrium bond distance is 1.0977 angstroms, and the dissociation energy is 9.79 eV. For a zero-point vibrational energy of 0.146 eV, what is the value of *a* (in angstroms)? The atomic weight of N is 14.0067 grams per mole (2.33×10^{-26} kg per atom) .

P7. The hyperfine splitting of the ground state in hydrogen is due to the interaction of electron spin $\vec{s}$ and the proton spin $\vec{i}$, and can be described by the interaction Hamiltonian,

$$H_{\text{int}} = A\delta(\vec{r})\left(\vec{s}\cdot\vec{i}\right)$$

where the constant *A* depends on the electron and proton magnetic moments and $\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$ is a delta-function in three dimensions. Using the hydrogen wave functions provided in the table, obtain an expression for the hyperfine splitting in terms of *A* and the Bohr radius a_0, to lowest order in H_{int} .

P8. Consider a two-port interferometer with 50/50 beam splitters, each arm having a path length L. The optical path for one arm can be made longer through a phase shift θ. A single photon of frequency ω is prepared and injected into the input port. If one of the two mirrors is only partially reflecting, with a reflection probability (reflectance) R $\leq$ 1 as indicated in the figure below, what is the probability of detecting the photon at the denoted output-port versus θ and R?



P9. A particle with mass *m* is trapped in the ground state of a harmonic oscillator well described by the potential energy function $U_1(x) = \frac{1}{2}kx^2$. *Suddenly* the particle is subjected to a constant force F, so that the new potential well is given by $U_2(x) = \frac{1}{2}kx^2 - Fx$. What is the probability that the particle will be in the ground state of the new well?

P10. The eigenstates $|n\rangle$ of the harmonic oscillator Hamiltonian $H = \hbar\omega\left(b^\dagger b + 1/2\right)$ are labeled by the quantum number n, with energy eigenvalue $E_n = \hbar\omega\left(n + 1/2\right)$. A coherent state,

$$|\lambda\rangle = e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle$$

is a superposition of these eigenstates, weighted respectively by $\frac{\lambda^n}{\sqrt{n!}}$, where λ is any complex number. Coherent states have the property that they are eigenstates of the lowering-operator, i.e., $b|\lambda\rangle = \lambda|\lambda\rangle$. Consider the time evolution of a coherent state. If the initial state $|\psi(0)\rangle = |\lambda\rangle$, where $\lambda = e^{i\phi}$ show that $|\psi(t)\rangle = e^{-\frac{i}{2}\omega t}|\lambda'\rangle$ where $\lambda' = e^{-i(\omega t - \phi)}$. Use this result to show that the expectation value of the position evolves according to $\langle x(t)\rangle = x_0\cos\left(\omega t - \phi\right)$, where $x_0 = \sqrt{\frac{2m}{\hbar\omega}}$.

Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy
University of New Mexico

January 2016

Instructions:

- *the exam consists of 10 problems, 10 points each;*
- *partial credit will be given if merited;*
- *total time is 3 hours.*

Table of Constants and Conversion factors

Quantity	Symbol	$\approx$ value
speed of light	c	3.00×10^8 m/s
Planck	h	6.63×10^{-34} J·s
electron charge	e	1.60×10^{-19} C
e mass	m_e	0.511 Mev/c^2 = 9.11×10^{-31} kg
p mass	m_p	938 Mev/c^2 = 1.67×10^{-27} kg
amu	u	931.5 Mev/c^2 = 1.66×10^{-27} kg
fine structure	α	1/137
Boltzmann	k_B	1.38×10^{-23} J/K
Bohr radius	$a_0 = \hbar/(m_e c\alpha)$	0.53×10^{-10} m
conversion constant	$\hbar c$	200 eV·nm = 200 MeV·fm
conversion constant	k_BT@ 300K	1/40 eV
conversion constant	1eV	1.60×10^{-19} J

Formulas

Pauli spin matricies:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Rotation of spinor about $\hat{n}$ direction by an angle ϕ:

$$\hat{R}(\phi\hat{n}) = \cos\left(\frac{\phi}{2}\right)\hat{I} - i\vec{\sigma}\cdot\hat{n}\sin\left(\frac{\phi}{2}\right)$$

Angular momentum raising,lowering operators

$$\hat{J}_\pm \left|j,m\right\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\left|j,m\pm 1\right\rangle$$

H.O. lowering operator

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right)$$

1. The carbon monoxide molecule absorbs radiation at a wavelength of 2.6 millimeters, corresponding to the rotational excitation of the molecule from $\ell = 0$ to $\ell = 1$. Calculate the molecular bond length.
2. Consider a particle of mass m in a time independent but complex potential $V(x) = V_0(x) - i\Gamma$ where Γ is independent of both space and time. Show that in this case the probability density is not conserved and find its time dependence.
3. Consider an electron bound in a harmonic oscillator potential with a linear perturbation.
$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \epsilon\frac{\hat{x}}{x_0}$$
where $x_0 = \sqrt{\hbar/m\omega}$ and $\epsilon << \hbar\omega$. What is the correction to the ground state energy to lowest non-vanishing order in perturbation theory?
4. Consider a particle in a harmonic oscillator potential. For an initial state of the particle given by a superposition of energy eigenstates,
$$|\Psi(x,0)\rangle = \frac{1}{\sqrt{2}}(|n\rangle + |n+1\rangle).$$
Calculate $\langle x\rangle$ as a function of time.
5. A particle of mass m moves on a flat ring of radius a but is otherwise free to move. Taking the symmetry axis of the ring to be the $\hat{z}$ axis, write the Hamiltonian in position space. Find the energy eigenstate position space wave functions and energy eigenvalues. What are the conserved physical quantities? How are the degeneracies related to the conserved quantities?
6. Consider the Ammonia molecule NH_3. The three hydrogens lie in a plane and form an isosceles triangle with the nitrogen along an axis perpendicular to the plane. The position-space wave function for the nitrogen moving in the potential of the three hydrogens has two linearly independent states: $|1\rangle$ and $|2\rangle$ corresponding to N above and below the plane of the hydrogens. (see Figure) The Hamiltonian in this basis has the form
$$[H] = \begin{bmatrix} E_0 & -A \\ -A & E_0 \end{bmatrix}$$
(a) If the system is in state $|1\rangle$ at $t = 0$, what is the probability to find the system in state $|1\rangle$ at time t?
(b) What is the physical interpretation of the off-diagonal elements?

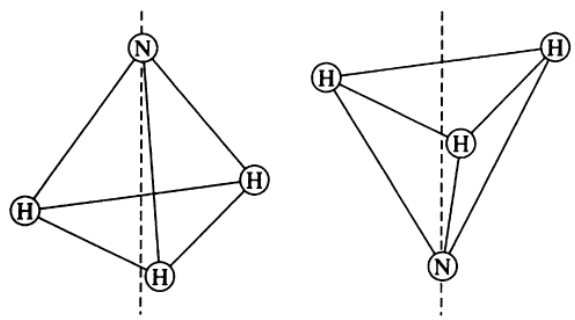


Figure 1: States $|1\rangle$ (left) and $|2\rangle$ (right) of the ammonia molecule.

7. A particle with low momentum $\hbar k$ is scattered by a hard sphere of radius a for which we can take the potential to be

$$V(r) = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases}$$

where r is the distance from the center of the sphere.
(a) The radial wave function for the $\ell = 0$ partial wave is of the form $u(r) = rR(r) = c\sin(kr + \delta_0)$ where δ_0 is the zeroth partial wave phase shift and c is a constant. Sketch $u(r)$ for $0 < r < \infty$, clearly indicating $r = a$. Determine the phase shift δ_0.
(b) What is the $k \to 0$ total cross section limit? Compare to the quantum to the classical result Recall the total cross section is

$$\sigma = \frac{4\pi}{k^2} \sum_\ell (2\ell + 1) \sin^2 \delta_\ell$$

where δ_ℓ is the ℓth partial wave phase shift.

8. Consider two electrons produced in an entangled spin state with total spin $S = 0$ (spin singlet state). What is the probability to measure one electron with spin-up along the direction $\hat{a}$ and the other electron with spin-up along the direction $\hat{b}$, where these directions are arbitrary?

9. A hydrogen atom in the presence of a magnetic field will have an additional interaction (Zeeman effect)

$$\hat{H}_{Zeeman} = \frac{\mu_B}{\hbar}\vec{B}\cdot\left(\hat{\vec{L}} + 2\hat{\vec{S}}\right)$$

where $\mu_B = 5.8 \times 10^{-5}$eV/T is the Bohr magneton.
(a) What constitutes the weak field regime? Give an order of magnitude estimate of an upper limit on B, in Tesla, for the weak field approximation to be valid.
(b) In the weak field approximation, find the correction to each of the states of the ${}^2P_{\frac{1}{2}}$ doublet. The explicit form these states is:

$$|\frac{1}{2},\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}\,|1,1\rangle\,|\frac{1}{2},-\frac{1}{2}\rangle - \sqrt{\frac{1}{3}}\,|1,0\rangle\,|\frac{1}{2},\frac{1}{2}\rangle$$

$$|\frac{1}{2},-\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}\,|1,0\rangle\,|\frac{1}{2},-\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|1,-1\rangle\,|\frac{1}{2},\frac{1}{2}\rangle$$

where the notation refers to $|j, m_j\rangle$ (total angular momentum quantum number j) on the left hand side and $|1, m_\ell\rangle\,|\frac{1}{2}, m_s\rangle$ are the product of orbital and spin wave functions on the right hand side.

10. Two identical, non-interacting spin-1/2 particles of mass m are in the same one dimensional harmonic oscillator potential.

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_1^2 + \frac{\hat{p}_2^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_2^2$$

a) Determine the ground state(s) in terms of harmonic oscillator states $|n\rangle_i$ and spin $|1/2, m\rangle_i$ where i is the particle label. What is the energy and degeneracy of the ground state?
b) Determine the first excited state(s). What is the energy and degeneracy of the first excited state?