Preliminary Examination: Thermodynamics and Statistical Mechanics

Department of Physics & Astronomy
University of New Mexico

Fall 2018

Instructions:

- This exam consists of 10 problems with 10 points each.
- Read all 10 problems before you begin to solve any problem, and solve the problems that seem easiest to you first. Spend your time wisely. If you are stuck on one problem, move on to the next one, and come back to it if you have time after you have solved all other problems.
- Show necessary intermediate steps in each solution. Partial credit will be given if merited.
- No textbook, personal notes or external help may be used other than what is provided by the proctor.
- This exam takes 3 hours.

Potentially Useful Information:

Physical constants and symbols:

k_{B}	Boltzmann's constant
$\hbar$	Planck's constant h divided by 2π

Formulas and relations:

$$\int_0^\infty x^{2n} e^{-\alpha x^2}\,\mathrm{d}x = \left(-\frac{\partial}{\partial\alpha}\right)^n \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \qquad \text{if } \mathrm{Re}(\alpha) > 0$$

$$\int_0^\infty x^{2n+1} e^{-\alpha x^2}\,\mathrm{d}x = \left(-\frac{\partial}{\partial\alpha}\right)^n \frac{1}{2\alpha} \qquad \text{if } \mathrm{Re}(\alpha) > 0$$

$$\ln N! \approx N \ln N - N \qquad \text{if } N \gg 1$$

$$1 + x + x^2 + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x}$$

Problem 1: The Hamiltonian for an one-dimensional, classical, anharmonic oscillator is given by

$$H = \frac{p^2}{2m} + \alpha x^8,$$

where p, m and x are the momentum, mass and position of the oscillator mass, respectively, and α is a positive constant. Use the equipartition theorem and the virial theorem

$$\left\langle p\frac{\partial H}{\partial p}\right\rangle = \left\langle x\frac{\partial H}{\partial x}\right\rangle$$

to determine the average energy of the oscillator in equilibrium with the environment at temperature T.

Problem 2: Consider a system of N non-interacting, spin-0 bosons held together in a confining potential which has only two quantum states with energies ε_1 and ε_2. Calculate the total energy of the system when it is in equilibrium at a temperature T.

Problem 3: Evaluate the partition function Z_1 of a single particle of a classical dilute gas which is confined to and in equilibrium with a three-dimensional container of volume V at temperature T. The particle has mass m. Express Z_1 as a ratio of V to a characteristic volume V_{c} of the particle.

Problem 4: A gas of $N \sim 10^{23}$ non-interacting spin-$1/2$ fermions of mass m and temperature $T = 0$ is confined to a cubic region of edge length L and volume $V = L^3$. Compute the total energy U and the pressure P of the gas.

Problem 5: Consider an ideal three-dimensional classical gas of a large number N of free non-interacting particles in equilibrium at temperature T. Each particle has energy proportional to the magnitude of its momentum. Find the free energy $F(T, V, N)$ of the gas.

Problem 6: A cup of coffee at temperature T_1 is mixed together with another two cups of coffee at temperature T_2 at constant pressure P in an insulated container. Assume that the specific heat of the coffee at constant pressure is temperature independent. Determine the final temperature of the mixture. State clearly any assumption(s) that you have made.

Problem 7: An isolated system consisting of N non-interacting distinguishable magnetic dipoles is in equilibrium in a uniform magnetic field of strength B. The energy of the system is $U = -N_1\mu B + (N - N_1)\mu B$ if there are N_1 dipoles aligned with the magnetic field (and the rest of the dipoles are against the field), where μ is a constant. Obtain an expression for the temperature of the system as a function of N, N_1 and B.

Problem 8: Calculate the temperature dependence of the entropy of a collection of a large number N of non-interacting distinguishable two-level systems (for instance atoms). Assume that the energy difference between the two levels is Δ.

Problem 9: The Helmholtz free energy of N molecules of a certain gas at temperature T and volume V is given by

$$F = -Nk_{\mathrm{B}}T\ln(V-b) - \frac{a}{V},$$

where a and b are constants. Find the pressure equation of state $P(T,V)$ of the gas. Explain the physical origins of the differences between the pressure equation of state of this gas and that of the ideal gas.

Problem 10: Compute the energy U and heat capacity C of a one-dimensional harmonic oscillator of frequency ω which is in equilibrium with a thermal bath of temperature T. The energy spectrum of the oscillator is $\varepsilon_n = (n+1/2)\hbar\omega$, where $n = 0, 1, 2, \ldots$ Determine the temperature dependence of $C(T)$ in the high and low temperature limits.