Preliminary Examination: Thermodynamics and Statistical Mechanics

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Instructions:

- This exam consists of 10 problems with 10 points each.
- Read all 10 problems before you begin to solve any problem, and solve the problems that seem easiest to you first. Spend your time wisely. If you are stuck on one problem, move on to the next one, and come back to it if you have time after you have solved all other problems.
- Show necessary intermediate steps in each solution. Partial credit will be given if merited.
- No textbook, personal notes or external help may be used other than what is provided by the proctor.
- This exam takes 3 hours.

Potentially Useful Information:

Physical constants and symbols:

k_B	Boltzmann's constant
$\hbar$	Planck's constant h divided by 2π
m_e	mass of the electron
e	electric charge of the proton

Formulas and relations:

Occupancy of a quantum state:

$$\bar{n} = \frac{1}{e^{(\varepsilon-\mu)/k_B T} \pm 1},$$

where ε, μ and T are the energy, chemical potential and temperature, respectively, and the plus/minus sign is for the Fermi-Dirac/Bose-Einstein statistics.

$$\int_0^\infty x^{2n} e^{-\alpha x^2}\,\mathrm{d}x = \left(-\frac{\partial}{\partial\alpha}\right)^n \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \qquad \text{if } \mathrm{Re}(\alpha) > 0$$

$$\int_0^\infty x^{2n+1} e^{-\alpha x^2}\,\mathrm{d}x = \left(-\frac{\partial}{\partial\alpha}\right)^n \frac{1}{2\alpha} \qquad \text{if } \mathrm{Re}(\alpha) > 0$$

$$\ln N! \approx N \ln N - N \qquad \text{if } N \gg 1$$

$$1 + x + x^2 + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x}$$

Problem 1: A gas of $N \gg 1$ non-interacting molecules is confined to a cubic box of volume V. Find an expression of the pressure of the gas as a function of N, V and the average translational kinetic energy of a gas molecule $\frac{1}{2}m\overline{v^2}$. The expression you find must also be true for noninteracting gases which do not obey the ideal gas law, i.e. fermi or bose gases.

Problem 2: Assume that the atoms in a solid are confined in one direction by a one-dimensional potential of the form

$$V(x) = ax^2 - bx^3 + \cdots ,$$

where a and b are constant with b being small in some appropriate sense, and x is the small displacement of the atom from its equilibrium position. Use this model to show that the expansion of the solid is proportional to the temperature T and the anharmonicity coefficient b. Assume the temperature is high enough that the summation over quantum states can be replaced by a phase space integral.

Problem 3: Calculate the translational part of the partition function of a single free particle of mass m confined to a container of volume V at temperature T. Show that the result has the suggestive form of the ratio of the container volume V to another volume characteristic of the particle at temperature T which involves Planck's constant h and the mass of the particle.

Problem 4: Electronic properties of many metals at room temperature can be understood through a model in which the conduction electrons are treated as a gas of non-interacting, non-relativistic, indistinguishable spin-1/2 particles of mass m_e at a very low temperature. Show that the typical speed of the fastest conduction electrons in a metal is

$$v_{\max} = (3\pi^2)^{1/3}\frac{\hbar n_e^{1/3}}{m_e},$$

where n_e is the number of conduction electrons per unit volume.

Problem 5: An isolated, cubic box of volume V_0 is divided into two equal parts. Initially one half of the box is occupied by an ideal gas of N molecules at temperature T_0. Suppose that the partition between the two halves of the box is suddenly lifted. Find the final temperature and the change in entropy ΔS of the gas after it has reached equilibrium again.

Problem 6: Blackbody radiation may be considered as a three-dimensional photon gas in equilibrium at a temperature T. Show that the energy per unit volume of this radiation is proportional to T^4.

Problem 7: An isolated one-dimensional system consisting of N non-interacting distinguishable spin-$1/2$ particles of magnetic moment μ is in equilibrium in a uniform magnetic field of strength B. The energy of one particle is $-\mu B$ if its spin is aligned with the magnetic field (spin-up) and μB if it is anti-aligned (spin-down). Obtain an expression for the temperature of the system as a function of N, B, and the average number of spin-up particles $\overline{N_\uparrow}$.

Problem 8: A one-dimensional harmonic oscillator of (angular) frequency ω has energy levels $\varepsilon_n = (n + 1/2)\hbar\omega$ ($n = 0, 1, 2, \ldots$). Calculate its Helmholtz free energy at temperature T.

Problem 9: A system consists of N non-interacting, distinguishable particles each of which can be in one of the two states with energies 0 and ε, respectively. Show that the relative fluctuation $\Delta E/\overline{E}$ of the system in equilibrium is proportional to $1/\sqrt{N}$, where $\overline{E}$ and $\Delta E = (\overline{E^2} - \overline{E}^2)^{1/2}$ are the average energy and the r.m.s. deviation of the energy of the whole system, respectively.

Problem 10: The equation of state and the internal energy of a gas of n moles of diatomic molecules at pressure P and temperature T are given by $PV = nRT$ and $U = \frac{5}{2}nRT$, respectively. Derive the specific heat at constant volume c_V of the gas. Show that the specific heat at constant pressure of the gas is $c_P = c_V + R$.