Preliminary Examination: Thermodynamics and Statistical Mechanics

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Instructions:

- The exam consists of 10 short-answer problems (10 points each).
- Where possible, show all work; partial credit will be given if merited.
- Personal notes on two sides of an 8×11 page are allowed.
- Total time: 3 hours.

P1: The ideal gas law $PV = nRT$ states the relationship between the pressure P exerted by a gas on the walls of the container of volume V to the temperature T of the gas, n being the number of moles and R the universal gas constant. Consider collisions of the gas molecules with the walls of the container and derive under reasonable assumptions the relationship

$$\frac{1}{2}mv^2 = \frac{3}{2}k_B T$$

between the translational kinetic energy of a molecule of the gas and the temperature of the gas. Carefully explain the assumptions you make during the derivation and show what the connection is between the Boltzmann constant k_B and the gas constant R.

P2: Develop a quantitative model of linear thermal expansion by assuming that atoms in a solid are confined in one direction by a one-dimensional potential energy function, that one may analyze the problem as involving a set of single particles each with a small displacement x from equilibrium, the confining potential being of the form $V(x) = ax^2 - bx^3 + \ldots$, b being small (in some appropriate sense), and using the Maxwell-Boltzmann distribution function to derive an expression for $\langle x \rangle$. Show that the expansion of the solid is linearly proportional to the temperature and to the anharmonicity coefficient b. You might find of use the integral

$$\int_{-\infty}^{\infty} e^{-mx^2} dx = \sqrt{\pi/m}.$$

P3: Evaluate the partition function of a free particle of mass m confined to, and in equilibrium with, a 3-dimensional container of volume V at temperature T and express it as the ratio of the container volume V to another volume V_c characteristic of the particle. State clearly the physical significance of V_c and comment on its dependence on the mass of the particle and the temperature. *Hints*: One method of evaluation is classical in approach and involves an integration over the coordinate x and momentum p, followed by the introduction of a factor $1/h^3$, where h is Planck's constant, to make the partition function dimensionless. Another method is quantum mechanical and involves a summation over the free particle states, approximating the summation by an integration. Choose whichever you prefer. The characteristic volume V_c is often called the thermal de Broglie volume of the particle. The Gaussian integral in problem P2 may be of use to you here.

P4: A gas of $N \sim 10^{23}$ non-interacting spin-1/2 fermions of mass m and temperature $T = 0$ is confined to a cubic region of edge length L and volume $V = L^3$. Find an expression for the largest occupied single particle energy ε_f (the Fermi energy) and express it as a function of the gas particle density $\rho = N/V$. Compute the total internal energy U of the gas, and use this to compute the pressure exerted by the gas on the container that holds it.

P5: Consider a 3-dimensional system of classical spins that do not interact among themselves but interact with a magnetic field B. If the angle θ between the moment and the field is *larger than* 45 *degrees* the energy U of a spin (magnetic moment μ) is zero. If the angle is *smaller than* 45 *degrees*, the energy is given by

$$U = -\mu B \cos\theta.$$

Calculate the partition function. Use it to calculate and sketch the B-dependence of the magnetization.

P6: Einstein's theory of the specific heat of an insulator relies on envisaging the solid as a collection of identical independent oscillators each of frequency ω and computing the average energy of each oscillator to be given by

$$\bar{\epsilon} = \left(\bar{n} + \frac{1}{2}\right)\hbar\omega$$

where $\bar{n} = \left(e^{\hbar\omega/k_B T} - 1\right)^{-1}$. Consider a similar situation with the difference that each oscillator is replaced by a two-level system, the energy difference between the two levels being Δ. Calculate the specific heat, and show how its behavior at low and high temperatures would differ from that of the Einstein model. Indicate also what you mean by high and low (temperature.)

P7: State the Fermi-Dirac distribution, the Bose-Einstein distribution, and the Maxwell-Boltzmann distribution describing the occupation number of a fixed number N of particles each with a given energy ϵ in terms of the chemical potential μ and temperature T. The crucial ratio in these expressions is $(\epsilon - \mu)/k_B T$. Consider the argument that the ratio goes to zero as the temperature becomes very large and that consequently the Fermi-Dirac distribution becomes classical in the *small* rather than large temperature limit? What is wrong in this argument? Explain.

P8: The underlying equations of mechanics are reversible in time. Yet every day behavior we observe is irreversible. Briefly explain how you understand the resolution of this paradox.

P9: Which of the two statements (if either) is correct and under what physical conditions? If neither is correct, write a corrected version of each and if the statements or slight modifications are valid under other physical conditions point it out carefully. A legalistic answer will not give you credit. Treat this question as an opportunity to show the examiners that you understand the relevant physics:

Statement 1: In thermal equilibrium, a large enough system occupies its energy states of energy E with equal probability.

Statement 2: In thermal equilibrium, a large enough system occupies its energy states of energy E with probability proportional to the factor $\exp(-E/k_BT)$ where T is the temperature.

P10: Planck's radiation law states that the average energy density per unit volume E_ν in a black body at frequency ν is given by

$$\frac{8\pi h(\nu/c)^3}{e^{h\nu/k_BT}-1}.$$

For 0.75 of the credit of this problem, state (with clear explanation) how this expression would be modified in Flatland (2-dimensional universe.) For full credit derive an expression for that case.