Department of Physics and Astronomy, University of New Mexico

Quantum Mechanics Preliminary Examination

Spring 2019

Instructions:

- This exam consists of 10 problems with 10 points each.
- Read all 10 problems before you begin to solve any problem, and solve the problems that seem easiest to you first. Spend your time wisely. If you are stuck in one problem, move on to the next one, and come back to it if you have time after you have solved all other problems.
- Show necessary intermediate steps. Partial credit will be given if merited.
- No textbook, personal notes, or external help may be used other than what is provided by the proctor.
- This exam takes 3 hours.

Useful Constants, Formulas, and Relations:

- Expectation values for the Hydrogen atom:

$$\langle \frac{1}{r^3} \rangle_{n,l} = \frac{1}{a_0^3 n^3} \frac{1}{l(l+1/2)(l+1)} \,.$$

- Harmonic oscillator Hamiltonian in terms of the lowering a and raising $a^\dagger$ operators:

$$\hat{H} = \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \hbar\omega \,.$$

- Ground state wavefunction of a harmonic oscillator with mass m and frequency ω:

$$\psi_0(x) = \left(\frac{1}{\pi x_0^2} \right)^{1/4} \exp\left(-\frac{x^2}{2x_0^2} \right) \quad , \quad x_0 = \left(\frac{\hbar}{m\omega} \right)^{1/2} .$$

- Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

- Rotation operator about direction $\hat{n}$ by an angle θ for a spin-1/2 particle:

$$\hat{R}(\theta, \hat{n}) = \cos\left(\frac{\theta}{2}\right) \hat{I} - i(\hat{n} \cdot \vec{\sigma}) \sin\left(\frac{\theta}{2}\right) .$$

- The radial Schrödinger equation for a central potential $V(r)$:

$$-\frac{\hbar^2}{2m}\frac{d^2\phi(r)}{dr^2} + \left(\frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right)\phi(r) = E\phi(r) \quad , \quad \phi(r) = r\psi(r) .$$

- Gaussian integrals:

$$\int_{-\infty}^{+\infty} x^{2n} \exp(-ax^2) = \frac{(2n-1)!!}{2^n}\sqrt{\frac{\pi}{a^{2n+1}}} .$$

1-A particle with mass m is incident from the left at $x = -\infty$ on the following step potential:

$$V(x) = V_0 \ , \qquad x \geq 0$$
$$V(x) = 0 \ , \qquad x < 0 \, .$$

If the particle energy is $E = \frac{3}{2}V_0$, what is the probability that the particle will reflect from the step?

2-A a two-level system has the following Hamiltonian:

$$\hat{H} = iA \left(|2\rangle\langle 1| - |1\rangle\langle 2| \right) \, ,$$

where A is a real number. The system starts in the state $|2\rangle$ at $t = 0$. If the system is measured at a later time t, what is the probability that it will be found in the state $|1\rangle$?

3-The wavefunction for a particle moving freely in one dimension is given by:

$$\psi(x) = \left(\frac{1}{\sqrt{2\pi}\sigma} \right)^{1/2} \exp\left(-\frac{x^2}{4\sigma^2} \right) \, ,$$

where σ is a constant. Calculate the variance of the expected outcomes of a measurement of the particle's position Δx and its momentum Δp. Verify that $\Delta x \Delta p = \hbar/2$.

4-A harmonic oscillator with frequency ω and mass m is in its ground state. The mass of the oscillator is suddenly increased by a factor of 16. What is the probability that the new oscillator will be in its ground state?

5-Consider the Hamiltonian that governs the evolution of the spin of an electron in a constant magnetic field B in the x direction:

$$\hat{H} = -\frac{2\mu}{\hbar} B \hat{S}_{\mathrm{x}} \, .$$

The electron is in the spin-up state in the z direction at $t = 0$. Derive an expression for $\langle \hat{S}_{\mathrm{z}} \rangle$ as a function of time.

6-A harmonic oscillator with frequency ω and mass m is prepared in the coherent state

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,,$$

where α is a complex number and $|n\rangle$ satisfies $\hat{a}^\dagger \hat{a}|n\rangle = n|n\rangle$ ($\hat{a}$ and $\hat{a}^\dagger$ are the lowering and raising operators respectively). Show that $|\alpha\rangle$ is an eigenstate of a and find the corresponding eigenvalue. If at the initial time $|\psi(0)\rangle = |\alpha\rangle$, find $|\psi(t)\rangle$.

7-Two electrons are prepared in an entangled spin state with total spin $S = 0$. Find the probability to measure one electron with spin-up in the direction of x and the other with spin-down in the y direction.

8-Two non-interacting spinless particles with the same mass m are confined to a one-dimensional infinite square well potential of width a. What is the ground state $|\psi_{\text{ground}}\rangle$ and the first excited state $|\psi_{\text{excite}}\rangle$ of this system? If now the particles are electrons, and ignoring interactions between the particles, find $|\psi_{\text{ground}}\rangle$ and $|\psi_{\text{excite}}\rangle$ (including both spin and motional components).

9-The spin-orbit coupling of the Hydrogen atom is described by the interaction Hamiltonian

$$\hat{H}_{\text{so}} = (1.77 \times 10^{-10} \text{ eV}) \left(\frac{a_0}{r}\right)^3 \frac{\vec{\hat{S}} \cdot \vec{\hat{L}}}{\hbar^2} \,,$$

where a_0 is the Bohr radius. Find the energy levels of the $2p$ state (expressed in eV) due to the spin-orbit coupling to the first order in perturbation theory.

10-Consider a particle with mass m in a three-dimensional potential well is described by the following Hamiltonian:

$$\begin{aligned} V(r) &= -V_0 \qquad r < a \\ V(r) &= 0 \qquad\quad r \geq a \,, \end{aligned}$$

where $V_0 > 0$. For given V_0 and m, find the minimum value of a in order to have at least one bound state.