Department of Physics and Astronomy, University of New Mexico

Quantum Mechanics Preliminary Examination

Spring 2018

Instructions:

- You should attempt all 10 problems (10 points each).
- Partial credit will be given if merited.
- NO cheat sheets are allowed.
- Total time: 3 hours.

Useful Constants, Formulas, and Relations:

- 1 eV $= 1.6 \times 10^{-19}$ J.
- Planck's constant: $h = 6.6 \times 10^{-34}$ Js.
- Bohr radius: $a_0 = 0.53 \times 10^{-10}$ m.
- Expectation values for the Hydrogen atom::

$$\langle \frac{1}{r^3} \rangle_{n,l} = \frac{1}{a_0^3 n^3} \frac{1}{l(l+1/2)(l+1)} .$$

- Harmonic oscillator lowering opertator:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p}_x \right) .$$

- Ground state wavefunction of a harmonic oscillator with mass m and potential $V(x) = \frac{1}{2}kx^2$:

$$\psi_0(x) = \left(\frac{1}{\pi x_0^2} \right)^{1/4} \exp\left(-\frac{x^2}{2x_0^2} \right) \quad , \quad x_0 = \left(\frac{\hbar}{\sqrt{km}} \right)^{1/2} .$$

- Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

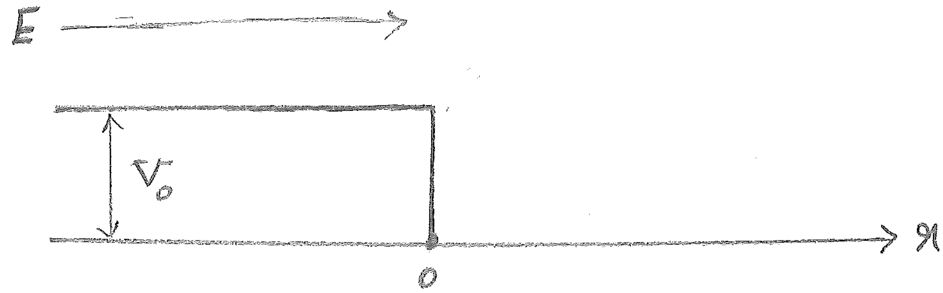
- Rotation operator about direction $\hat{n}$ by an angle θ for a spin-1/2 particle:

$$\hat{R}(\phi\hat{n}) = \cos\left(\frac{\theta}{2}\right)\hat{I} - i\vec{\sigma}\cdot\hat{n}\,\sin\left(\frac{\theta}{2}\right) .$$

- The radial Schrödinger equation for a central potential $V(r)$:

$$-\frac{\hbar^2}{2m}\frac{d^2\phi(r)}{dr^2} + \left(\frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right)\phi(r) = E\phi(r) \quad , \quad \phi(r) = r\psi(r) .$$

1– At time $t = 0$ a particle with mass m is incident from the left on a step potential shown in the figure. If the particle has energy $E = \frac{3}{2}V_0$, what is the probability that it will be on the right as $t \to \infty$?



2– Consider a particle confined by an infinite square well $0 < x < L$ in one dimension. The paticle is initially in the state

$$\psi(x,0) = \frac{1}{\sqrt{L}}\left(\sin\frac{\pi x}{L} - i\sin\frac{2\pi x}{L}\right) .$$

Calculate probability flux at $t = 0$. In which direction will the particle most likely be moving at $x = L/2$?

3- Consider a two-state system with the following Hamiltonian:

$$\hat{H} = V(|1\rangle\langle 1| + |2\rangle\langle 2|) + \Delta(|1\rangle\langle 2| + |2\rangle\langle 1|)\,,$$

where V and Δ are Hamiltonian matrix elements. The system starts in the state $|2\rangle$ at $t = 0$. What is the probability that the system will be in state $|1\rangle$ at a later time t?

4– Consider a simple harmonic oscillator in one dimenstion. Suppose that the oscillator is initially in a superposition of two energy eigenstates $|\psi\rangle = (|n\rangle - |n-1\rangle)/\sqrt{2}$. Calculate the expectation value of the momentum as a function of time.

5- The nuclei of many large atoms are aspherical and their rotational motion is described by the Hamiltonian for an axially symmetric rotator:

$$\hat{H} = \frac{\hat{L}_x^2 + \hat{L}_z^2}{2I_1} + \frac{\hat{L}_y^2}{2I_2}\,,$$

where $I_2 < I_1$. Find the ratio I_1/I_2 such that the energy levels of the $l = 3$ and $l = 4$ multiplets begin to overlap.

6- A particle with mass m is in the ground state of a two-dimensional harmonic oscillator with the potential energy $V = \frac{1}{2}k(x^2 + y^2)$. Suddenly a constant force F in the positive x direction is turned on. What is the probability that the particle will be in the ground state of the new potential?

7– Consider two electrons produced in an entangled spin state with total spin $S = 0$ (spin singlet state). What is the probability to measure one electron with spin-up in the z direction and the other electron with spin-down in the y direction?

8- Two non-interacting identical spin-1/2 particles are confined within a three-dimensional cubic box of side L. Write down the total wavefunction (including both space and spin) for the ground state of this two-particle system.

9– Consider a particle of mass m that is inside a hard sphere potential well with radius R. Derive the wavefunction and the energy of the ground state of this particle.

10– The spin-orbit coupling for the Hydrogen atom is described by the interaction Hamiltonian

$$H_{so} = (1.77 \times 10^{-10}\ \text{eV}) \left(\frac{a_0}{r}\right)^3 \frac{\vec{S} \cdot \vec{L}}{\hbar^2},$$

where a_0 is the Bohr radius. Find the energy levels of all of the $4p$ states (expressed in eV) due to the spin-orbit coupling to the first order in perturbation theory.