Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy

University of New Mexico

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Instructions:
• The exam consists of 10 problems (10 points each).
• Where possible, show all work; partial credit will be given if merited.
• Personal notes on two sides of an 8×11 page are allowed.
• Total time: 3 hours.

Useful Information:
1. Pauli sigma matrices: \( \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)
2. \( \int_0^{\pi/2} dx \sin(2x) \sin(x) = -2/3 \)
3. For a particle with mass \( m \) moving under the influence of an attractive one-dimensional square-well potential, \( V(x) = -V_0 \) for \( |x| \leq a \) and \( V(x) = 0 \) otherwise, the energy eigenvalues for the bound states are determined from the transcendental equation

\[
\frac{\sqrt{\lambda - y^2}}{y} = \begin{cases} 
\tan y & \text{(even parity)} \\
-\cot y & \text{(odd parity)} 
\end{cases}
\]

where \( \lambda = 2ma^2V_0/\hbar^2 \) and \( y = qa \) with \( q = \sqrt{\frac{2m}{\hbar^2}} (V_0 - E) \).
4. Some hydrogen wave functions:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \ell )</th>
<th>( m )</th>
<th>( R_{nm} )</th>
<th>( Y_{\ell m} )</th>
<th>( \psi_{nm} = R_{nm} Y_{\ell m} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( 2 \left( \frac{1}{2a_0} \right)^{3/2} e^{-r/a_0} )</td>
<td>( \frac{1}{2\sqrt{\pi}} )</td>
<td>( \frac{1}{2\sqrt{\pi}} \left( \frac{1}{a_0} \right)^{3/2} e^{-r/a_0} )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>( \left( \frac{1}{2a_0} \right)^{3/2} \left( \frac{2 - \frac{\pi}{2}}{a_0} \right) e^{-r/2a_0} )</td>
<td>( 1/2\sqrt{\pi} )</td>
<td>( 1/4\sqrt{2\pi} \left( \frac{1}{a_0} \right)^{3/2} \left( \frac{2 - \frac{\pi}{2}}{a_0} \right) e^{-r/2a_0} )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>( \left( \frac{1}{2a_0} \right)^{3/2} \frac{1}{\sqrt{3}} e^{-r/2a_0} )</td>
<td>( \frac{1}{2\sqrt{3}} \cos \theta )</td>
<td>( 1/4\sqrt{2\pi} \left( \frac{1}{a_0} \right)^{3/2} \frac{1}{\sqrt{3}} \frac{1}{a_0} e^{-r/2a_0} \cos \theta )</td>
</tr>
<tr>
<td>2</td>
<td>\pm 1</td>
<td>\pm 1</td>
<td>( \left( \frac{1}{2a_0} \right)^{3/2} \frac{1}{\sqrt{3}} \frac{1}{a_0} e^{-r/2a_0} )</td>
<td>( \pm 1/2\sqrt{2\pi} \sin \theta e^{\pm \phi} )</td>
<td>( 1/8 \sqrt{\pi} \left( \frac{1}{a_0} \right)^{3/2} \frac{1}{\sqrt{3}} \frac{1}{a_0} e^{-r/2a_0} \sin \theta e^{\pm \phi} )</td>
</tr>
</tbody>
</table>
P1. A particle with mass \( m \) moving to the right with kinetic energy \( E \) encounters a step potential of height \( V \); as shown in the figure. What is the probability that the particle will be reflected if \( E = (4/3) V \)?

\[ \text{P2.} \text{ Consider a beam of electrons, to be fired a great distance, } L = 10^4 \text{ km, along the x axis. In momentum-space the electron wavefunction is a strongly peaked Gaussian at } \hbar k_0 \text{ with energy } \hbar \omega = \frac{(\hbar k_0)^2}{2m} = 10 \text{ eV. The real-space wavefunction evolves in time according to} \]

\[ \psi(x, t) = A \exp \left[ i (k_0 x - \omega t) \right] \exp \left[ -\frac{(x - \omega t)^2}{4 (\sigma^2 + \frac{1}{4} \omega^' t)} \right], \]

where \( A \) and \( \sigma \) are constants and \( \omega' = (\partial \omega / \partial k)|_{k=k_0} \) denotes differentiation of \( \omega \) with respect to \( k \).

If the initial uncertainty in the position of an electron is \( \Delta x = 1.0 \text{ mm}, \) approximately what will be \( \Delta x \) on arrival?

P3. The Hamiltonian for an anharmonic oscillator in one dimension is given by

\[ \hat{H} = \frac{\hat{p}^2}{2m} + g \hat{x}^4 \]

where \( g \) is positive. Use the uncertainty principle to estimate the ground state energy. Express your answer as a function of \( g, \hbar, \) and the mass \( m \).
P4. A particle with mass $m$ is initially in the ground state of a one-dimensional infinite square well with width $a$. If the square well is suddenly stretched to a new width $2a$, what is the probability that a measurement of the energy will find the particle to be in the ground state of the new square well?

P5. The motion of a particle with mass $m$ moving in one dimension is described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + mg\hat{x}.$$  

Obtain an expression for the expectation value of the position $\langle \hat{x}(t) \rangle$ in terms of the initial expectation values $\langle \hat{p}(0) \rangle$ and $\langle \hat{x}(0) \rangle$.

P6. The nucleus of a gold atom is found to be aspherical, for its rotational motion is described by the Hamiltonian for the axially symmetric rotator,

$$\hat{H} = \frac{\hat{L}_x^2}{2I_1} + \frac{\hat{L}_y^2}{2I_2}.$$  

where the moment of inertia $I_1 > I_2$. Sketch the splittings in the rotational energy spectrum for $\ell = 0, 1, \text{ and } 2$. 

[Diagram of a gold atom]
P7. The Hamiltonian for a harmonic oscillator is given by

\[ \hat{H} = \hbar \omega_0 \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \]

where the operators \( \hat{a} \) and \( \hat{a}^\dagger \) satisfy the commutation relation \( [\hat{a}, \hat{a}^\dagger] = 1 \).

Consider an eigenstate \( |n\rangle \) of \( \hat{H} \) with eigenvalue \( (n + \frac{1}{2}) \hbar \omega_0 \). Prove that the two states \( |\phi(-)\rangle = C(-) \hat{a} |n\rangle \) and \( |\phi(+)\rangle = C(+) \hat{a}^\dagger |n\rangle \) are also eigenstates of \( \hat{H} \), where \( C(-) \) and \( C(+) \) are normalization constants. Find their respective eigenvalues and normalization constants.

P8. In quasi one-dimensional conductors, the Coulomb repulsion between conduction electrons is screened out at large distances. All that remains is small phonon-mediated interaction. This interaction is attractive, and is often modeled by a square-well potential.

Consider two electrons in a carbon nanotube, each in the same spin state, their center of mass free to move in the \( x \) direction. Suppose that they are attracted to one another by the square-well interaction \( V(x_1 - x_2) = -V_0 \) for \( |x_1 - x_2| \leq a \) and \( V(x_1 - x_2) = 0 \) otherwise, where the argument \( x_1 - x_2 \) is the difference between the displacements \( x_1 \) and \( x_2 \) of electron 1 and electron 2 respectively. What is the minimum depth \( V_0 \) required for a bound state for the two electron system? You may want to make use of the square-well solution listed in the table.
P9. An electron initially in an eigenstate of $\hat{S}_x$ with eigenvalue $\hbar/2$ is placed in a uniform magnetic field $\vec{B} = B_0 \hat{z}$. At some later time $t$ a measurement is to be made of $\hat{S}_y$. What is the probability that a value $\hbar/2$ will be found?

P10. If the photon actually had a small mass $m^*$, electrons and protons would attract one another via a screened (Yukawa) potential of the form

$$U(r) = \frac{-e^2}{4\pi\epsilon_0 r} e^{-\gamma r},$$

where the screening parameter $\gamma = m^*c/\hbar$. This would show up as causing a shift in the hydrogen spectrum.

To first order in $\gamma$, calculate the shift in the energy $E_{0,0,0}$ of the hydrogen ground state. What is the first order shift in the energy $E_{n,\ell,m}$ for arbitrary quantum numbers $n, \ell, \text{and} ~ m$?