Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy
University of New Mexico

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Instructions:

•The exam consists of 10 problems (10 points each).
•Where possible, show all work; partial credit will be given if merited.
•Personal notes on two sides of an 8×11 page are allowed.
•Total time: 3 hours.

Useful Information:

1. Pauli sigma matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2. $\int_0^{\pi/2} dx \sin(2x)\sin(x) = -2/3$

3. For a particle with mass m moving under the influence of an attractive one-dimensional square-well potential, $V(x) = -V_0$ for $|x| \leq a$ and $V(x) = 0$ otherwise, the energy eigenvalues for the bound states are determined from the transcendental equation

$$\frac{\sqrt{\lambda - y^2}}{y} = \begin{cases} \tan y & \text{(even parity)} \\ -\cot y & \text{(odd parity)} \end{cases},$$

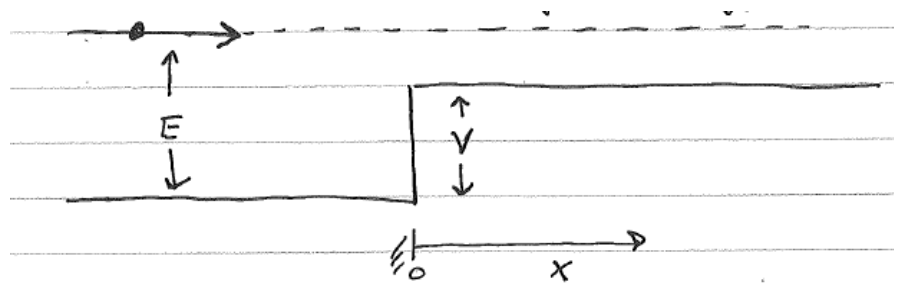
where $\lambda = 2ma^2V_0/\hbar^2$ and $y = qa$ with $q = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$.

4. Some hydrogen wave functions:

Table 1: Wave functions and their components

n	ℓ	m	$R_{n\ell}$	$Y_{\ell m}$	$\psi_{n\ell m} = R_{n\ell} Y_{\ell m}$
1	0	0	$2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$
2	0	0	$\left(\frac{1}{2a_0}\right)^{3/2}\left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2}\left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$
2	1	0	$\left(\frac{1}{2a_0}\right)^{3/2} \frac{1}{\sqrt{3}}\frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$	$\frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2} \frac{r}{a_0} e^{-r/2a_0}\cos\theta$
2	1	± 1	$\left(\frac{1}{2a_0}\right)^{3/2} \frac{1}{\sqrt{3}}\frac{r}{a_0} e^{-r/2a_0}$	$\pm\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{\pm i\phi}$	$\frac{1}{8}\sqrt{\frac{1}{\pi}}\left(\frac{1}{a_0}\right)^{3/2} \frac{r}{a_0} e^{-r/2a_0}\sin\theta e^{\pm i\phi}$

P1. A particle with mass m moving to the right with kinetic energy E encounters a step potential of height V, as shown in the figure. What is the probability that the particle will be reflected if $E = (4/3)\,V$?



P2. Consider a beam of electrons, to be fired a great distance, $L = 10^4$ km, along the x axis. In momentum-space the electron wavefunction is a strongly peaked Gaussian at $\hbar k_0$ with energy $\hbar\omega = \frac{(\hbar k_0)^2}{2m} = 10$ eV. The real-space wavefunction evolves in time according to

$$\psi(x,t) = A \exp\left[i\left(k_0 x - \omega t\right)\right] \exp\left[-\frac{(x-\omega' t)^2}{4\left(\sigma^2 + \frac{i}{2}\omega'' t\right)}\right],$$

where A and σ are constants and $\omega' = (\partial\omega/\partial k)|_{k=k_0}$ denotes differentiation of ω with respect to k .

If the initial uncertainty in the position of an electron is $\Delta x = 1.0$ mm, approximately what will be Δx on arrival?

P3. The Hamiltonian for an anharmonic oscillator in one dimension is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + g\hat{x}^4$$

where g is positive. Use the uncertainty principle to estimate the ground state energy. Express your answer as a function of g, $\hbar$, and the mass m.

P4. A particle with mass m is initially in the ground state of a one-dimensional infinite square well with width a. If the square well is suddenly stretched to a new width $2a$, what is the probability that a measurement of the energy will find the particle to be in the ground state of the new square well?

P5. The motion of a particle with mass m moving in one dimension is described by the Hamiltonian

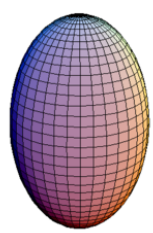
$$\hat{H} = \frac{\hat{p}^2}{2m} + mg\hat{x}.$$

Obtain an expression for the expectation value of the position $\langle \hat{x}(t) \rangle$ in terms of the initial expectation values $\langle \hat{p}(0) \rangle$ and $\langle \hat{x}(0) \rangle$.

P6. The nucleus of a gold atom is found to be aspherical, for its rotational motion is described by the Hamiltonian for the axially symmetric rotator,

$$\hat{H} = \frac{\hat{L}_x^2 + \hat{L}_y^2}{2I_1} + \frac{\hat{L}_z^2}{2I_2}$$

where the moment of inertia $I_1 > I_2$. Sketch the splittings in the rotational energy spectrum for $\ell = 0, 1,$ and 2.



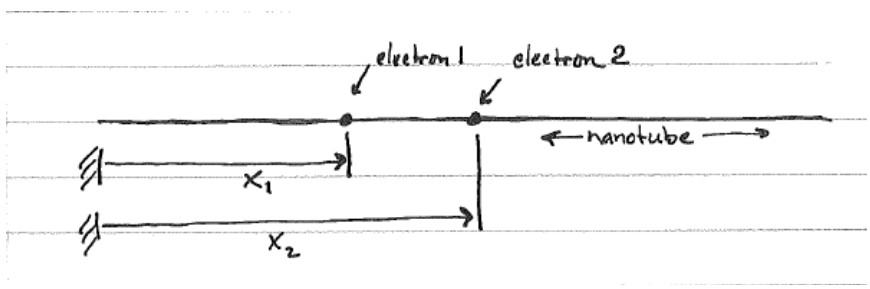
P7. The Hamiltonian for a harmonic oscillator is given by

$$\hat{H} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right),$$

where the operators $\hat{a}$ and $\hat{a}^\dagger$ satisfy the commutation relation $\left[\hat{a}, \hat{a}^\dagger\right] = 1$.

Consider an eigenstate $|n\rangle$ of $\hat{H}$ with eigenvalue $\left(n + \frac{1}{2}\right)\hbar\omega_0$. Prove that the two states $\left|\phi_{(-)}\right\rangle = C_{(-)}\hat{a}\,|n\rangle$ and $\left|\phi_{(+)}\right\rangle = C_{(+)}\hat{a}^\dagger\,|n\rangle$ are also eigenstates of $\hat{H}$, where $C_{(-)}$ and $C_{(+)}$ are normalization constants. Find their respective eigenvalues and normalization constants.

P8. In quasi one-dimensional conductors, the Coulomb repulsion between conduction electrons is screened out at large distances. All that remains is small phonon-mediated interaction. This interaction is attractive, and is often modeled by a square-well potential.



Consider two electrons in a carbon nanotube, each in the same spin state, their center of mass free to move in the x direction. Suppose that they are attracted to one another by the square-well interaction $V(x_1 - x_2) = -V_0$ for $|x_1 - x_2| \leq a$ and $V(x_1 - x_2) = 0$ otherwise, where the argument $x_1 - x_2$ is the difference between the displacements x_1 and x_2 of electron 1 and electron 2 respectively. What is the minimum depth V_0 required for a bound state for the two electron system? You may want to make use of the square-well solution listed in the table.

P9. An electron initially in an eigenstate of $\hat{S}_x$ with eigenvalue $\hbar/2$ is placed in a uniform magnetic field $\vec{B} = B_0\ \hat{z}$. At some later time t a measurement is to be made of $\hat{S}_y$. What is the probability that a value $\hbar/2$ will be found?

P10. If the photon actually had a small mass m^*, electrons and protons would attract one another via a screened (Yukawa) potential of the form

$$U(r) = \frac{-e^2}{4\pi\epsilon_0 r} e^{-\gamma r},$$

where the screening parameter $\gamma = m^* c/\hbar$. This would show up as causing a shift in the hydrogen spectrum.

To first order in γ, calculate the shift in the energy $E_{0,0,0}$ of the hydrogen ground state. What is the first order shift in the energy $E_{n,\ell,m}$ for arbitrary quantum numbers n, ℓ, and m?