Department of Physics and Astronomy, University of New Mexico

Quantum Mechanics Preliminary Examination

Spring 2009

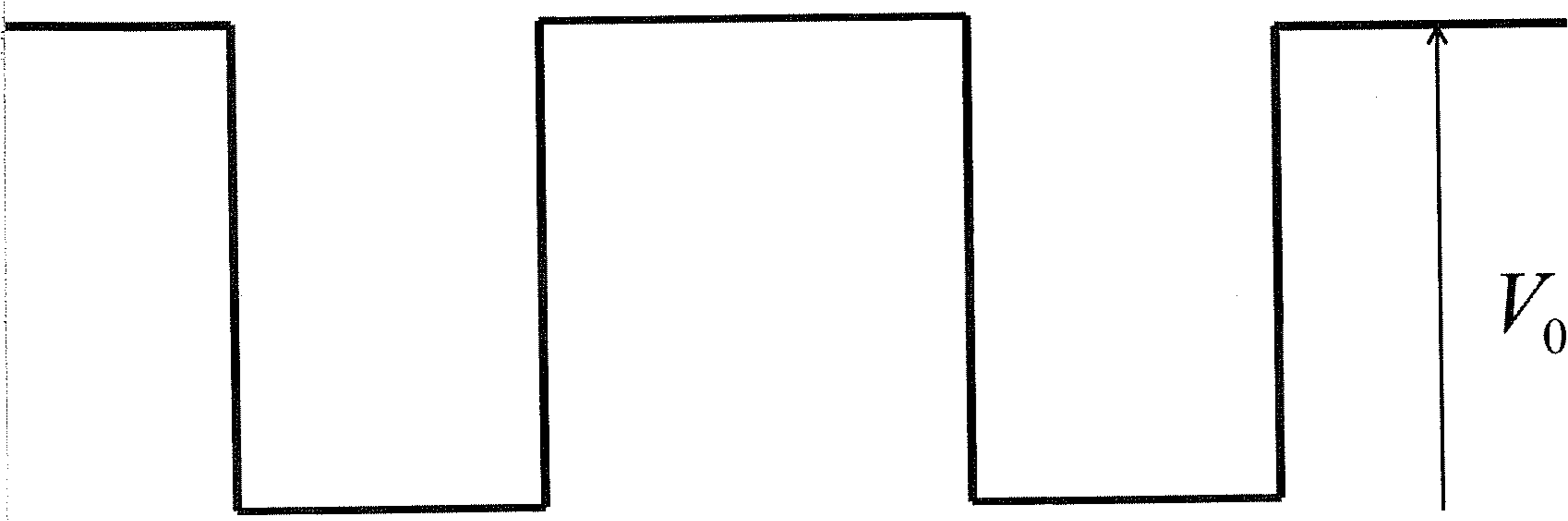
Instructions:

- The exam consists of 10 short-answer problems (10 points each).
- Partial credit will be given if merited.
- Personal notes on two sides of an $8\frac{1}{2}'' \times 11''$ page are allowed.
- Total time: 3 hours.

1– There is a symmetric double well potential as shown in the figure. Assume that this system has at least two bound states, namely the ground state with energy E_0 and the first excited state with energy E_1.

(a) Draw the wavefunctions for the ground state and the first excited state.

(b) What do the wavefunctions look like in the limit that $V_0 \to \infty$? What is the relationship between E_0 and E_1 in this limit?



2– Consider a free particle of mass m in one dimension. At $t = 0$ the wavefunction of the particle in position space is given by the following Gaussian wavepacket:

$$\psi(x) = \frac{1}{\pi^{1/4}\sqrt{a}}\exp[-\frac{(x-x_0)^2}{2a^2}].$$

In momentum space this wavefunction is given by:

$$\psi(p) = \frac{1}{\pi^{1/4}}\sqrt{\frac{2a}{\hbar}}\exp(-\frac{2a^2p^2}{\hbar^2}).$$

(a) Find $\bar{X},\ \bar{P},\ \Delta X,\ \Delta P$ at $t = 0$ (in terms of a, m and $\hbar$) and show that $\Delta X \Delta P = \hbar/2$.
Hint: helpful expression:

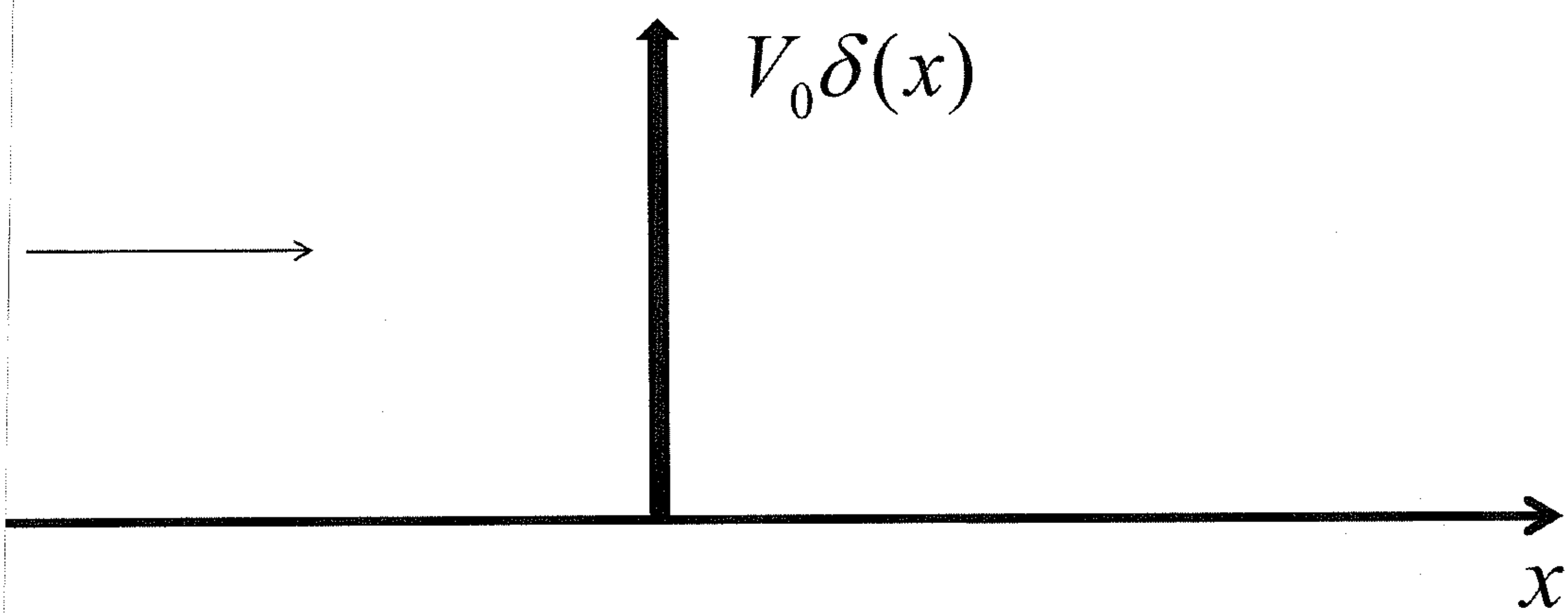
$$\int_0^\infty x^2 e^{-x^2/2} dx = \frac{\sqrt{\pi}}{2}.$$

(b) Describe in words what will happen to $\bar{X},\ \bar{P},\ \Delta X,\ \Delta P$ at $t > 0$.

3– A particle with mass m moves in one dimension and approaches a localized potential barrier $V = V_0\delta(x)$ from the left as shown in the figure. The wavefunction of the particle is given by (up to an overall normalization factor)

$$\psi(x) = e^{ikx} + Ae^{-ikx} \qquad x \le 0$$
$$\psi(x) = Be^{ikx} \qquad x > 0\,.$$

Find A and B in terms of V_0 and k.



4– The phenomenon of neutrino oscillations provides a solution to the solar neutrino puzzle. It can be explained by considering the time evolution of a quantum-mechanical system that has two energy eigenstates $|1\rangle$ and $|2\rangle$ with corresponding energy eigenvalues E_1 and E_2. The neutrino flavors correspond to the states $|a\rangle$ and $|b\rangle$ defined as follows:

$$|a\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}} \quad , \quad |b\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}.$$

The system starts in the state $|a\rangle$ at $t = 0$.

(a) Find the wavefunction of the system at $t > 0$.

(b) What is the probability $P_{a \to a}$ to find the system in the state $|a\rangle$ at time t? At which times $P_{a \to a}$ reaches its maximum value?

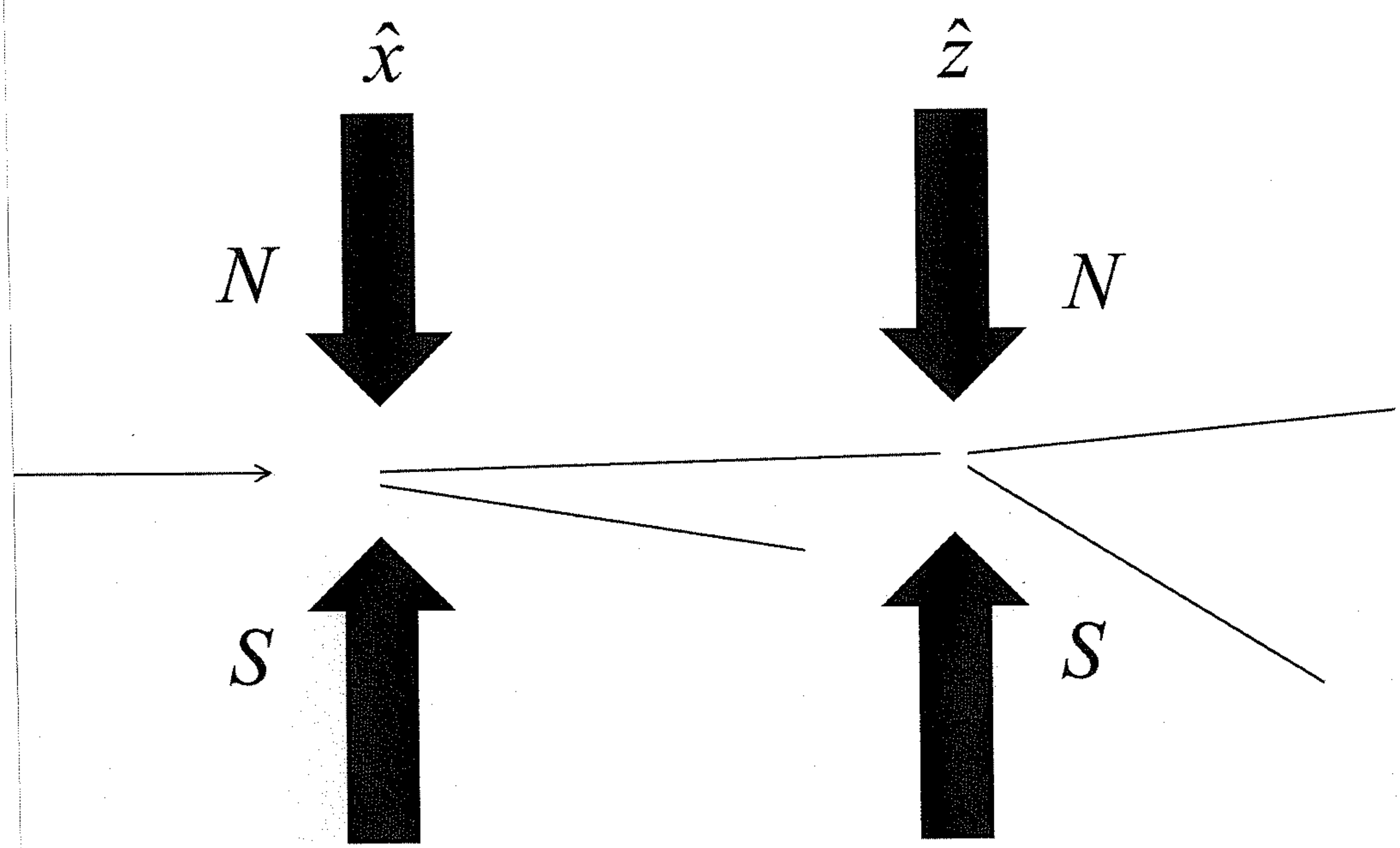
(c) What is the probability $P_{a \to b}$ to find the system in the state $|b\rangle$ at time t? At which times $P_{a \to b}$ reaches it maximum value?

5– An electron is in the spin-up state **along z** at $t = 0$. It then goes through two successive Stern-Gerlach apparatuses the second rotated by 90 degrees from the first as indicated in the figure.

(a) What is the probability for the electron to be found in the spin-up state **along x** after going through the first apparatus?

(b) After the electron is measured along the **x-axis** and found to be spin-up, it goes through the second apparatus. What is the probability for the electron to be found in the spin-down state **along z**?

If N electrons start in the spin-up state **along z** at $t = 0$, for large N what fraction will be found in the spin-down state **along z** after passing through the two Stern-Gerlach apparatuses?



6– Consider a system consisting of an electron and a positron in three-dimensional space of infinite volume including the center-of-mass and relative coordinates. Ignoring the spin of particles:

(a) Write the Hamiltonian for this system.

(b) What are all quantum numbers needed to specify a unique energy eigeenstate?

(c) Find the energy of the ground state.

7– A coherent state represents the closest quantum-mechanical wave-packet to a classical motion. It is constructed from the eigenstates of a harmonic oscillator as follows:

$$|\psi\rangle = \exp(-|c|^2/2) \sum_{n=0}^{\infty} \frac{c^n}{\sqrt{n!}} |n\rangle,$$

where c is an arbitrary complex number, $|0\rangle$ is the ground state and $\mathbf{a}^\dagger$ is the rasing operator. Derive the following properties of the state $|\psi\rangle$:

(a) $|\psi\rangle$ is an eigenstate of the lowering operator $\mathbf{a}$ with the eigenvalue c:

$$\mathbf{a}|\psi\rangle = c|\psi\rangle.$$

(b)

$$\bar{N} \equiv \langle\psi|(\mathbf{a}^\dagger \mathbf{a})|\psi\rangle = |c|^2.$$

(c)

$$(\Delta N)^2 \equiv \langle\psi|(\mathbf{a}^\dagger \mathbf{a})^2|\psi\rangle - \bar{N}^2 = |c|^2.$$

(d)

$$\frac{\Delta N}{\bar{N}} \to 0 \qquad \text{as } \bar{N} \to \infty.$$

8– Two identical non-interacting spin-1/2 particles of mass m are inside a cubic box with hard walls (i.e. infinite potential at the walls). The volume of the box is $V = L^3$. Write the wavefunction for the ground state of this system including the spatial and spin parts.

9– A system consists of two spin-1 particles and its Hamiltonian has no spin dependence.

(a) What is the degeneracy of each energy level of this system due to the spin degree of freedom?

(b) The Hamiltonian now has an additional term $H_S = a\vec{S}_1 \cdot \vec{S}_2$, where $\vec{S}_1$ and $\vec{S}_2$ are the spins of the two particles. Find the energy splitting among degenerate states in part (a) as a result of this interaction.

10– The Hamiltonian for a harmonic oscillator in two dimensions is given by:

$$H_0 = \frac{P_x^2 + P_y^2}{2m} + \frac{1}{2} m\omega^2 (x^2 + y^2). \tag{1}$$

(a) Find the energy eigenvalues of this oscillator. What is the energy and degeneracy of the first excited level?

(b) Now we add a term $H_1 = Axy$ to the potential ($A \ll m\omega^2$ is a constant). Find the energy eigenvalues and eigenstates of the first level in this case.