1. A spin-1/2 particle is in the state,

\[ |\psi\rangle = \frac{1}{\sqrt{3}} \{ | \uparrow z \rangle + i\sqrt{2} | \downarrow z \rangle \} , \]

where in this notation \( | \uparrow z \rangle \) labels the state that has spin-up with respect to the \( z \)-axis. Find the probability for a particle in this state to pass through an Stern-Gerlach device oriented along \(+x\) with spin-up followed by a Stern-Gerlach device oriented along \(+z\) with spin-down. Note that \( | \uparrow x \rangle = \frac{1}{\sqrt{2}} \{ | \uparrow z \rangle + | \downarrow z \rangle \} \).

2. A particle is incident from the left \((x < 0)\) on step potential,

\[
V(x) = \begin{cases} 
V_0, & x > 0 \\
0, & x < 0 
\end{cases}
\]

a) For \( E < V_0 \), sketch the wave function for all \( x \).

b) Let's call the wave function for \( x < 0 \) \( \Psi_- \). Then for \( E < V_0 \) \( \Psi_- \) must be of the form,

\[ \Psi_- = e^{ikx} + e^{i\delta} e^{-ikx} , \]

where \( \delta \) is a real constant. Why?

c) What is \( \delta \) in the limit \( V_0 >> E \)?

3. Consider a particle in a 1-D box \( 0 < x < a \) which at \( t = 0 \) is in the state,

\[
\psi(x) = \begin{cases} 
\sqrt{\frac{3}{a}} \left(2x/a\right), & 0 < x < a/2 \\
\sqrt{\frac{3}{a}} \left(2 - 2x/a\right), & a/2 < x < a ,
\end{cases}
\]

Determine the probabilities \( P_1, P_2 \) to measure the ground state energy \( E_1 \) and the first excited state energy \( E_2 \) at \( t = 0 \). (use \( \int_0^{\pi/2} u \sin(u) du = 1 \) )
4. Consider a system with orthonormal basis states $|1\rangle$ and $|2\rangle$. The Hamiltonian in this basis is given by,

$$\hat{H} = \begin{pmatrix} E_0 & -iA \\ iA & E_0 \end{pmatrix}$$

where $E_0$ and $A$ are real, positive constants. Find the energy eigenvalues and corresponding normalized energy eigenstates. Be clear as to which state goes with which eigenvalue.

5. Consider a spinless particle that is constrained to move on a circle of radius $R$ but free to move around the circle. In a cylindrical coordinate system with the circle in the $x$-$y$ plane, the wave function depends only on the azimuthal angle $\phi$.

   a) Write the Hamiltonian for this system.
   b) What are the energy eigenvalues and normalized eigenstates? Identify all good quantum numbers and any degeneracies.
   c) What is the uncertainty in $\phi$ for states of definite energy and angular momentum?

6. Consider a particle subject to the harmonic oscillator Hamiltonian,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m \omega^2 \hat{x}^2}{2}.$$ 

For an initial state of the particle given by a superposition of energy eigenstates,

$$|\Psi(x, 0)\rangle = \frac{1}{\sqrt{2}}(|n\rangle + |n + 1\rangle).$$

Calculate $\langle x \rangle$ as a function of time.

7. Consider a diatomic molecule as a rigid rotor with moment of inertia $I$ and magnetic moment $\vec{\mu} = -\mu_0 \vec{L}$, $\mu_0$ being a positive constant. The molecule is in a uniform magnetic field $\vec{B}$ along the $z$-axis.

   (a) What is the Hamiltonian for this system?
   (b) If at $t = 0$ the wave function is,

$$\Psi(0) = \frac{Y_{11} + Y_{10}}{\sqrt{2}},$$

calculate $\langle L_x \rangle$ as a function of time, using

$$\hat{L}_x = \frac{\hat{L}_+ + \hat{L}_-}{2}.$$
8. A particle moving in a central potential $V(r)$ has a Hamiltonian,

$$\hat{H} = \frac{1}{2m}\{\hat{P}_r^2 + \frac{\hat{L}^2}{r^2}\} + V(r),$$

where

$$\hat{P}_r = \frac{\hbar}{i} \frac{\partial}{\partial r}.$$

a) Show that

$$\Psi(r) = \frac{u(r)}{r} Y_{\ell m}(\theta, \phi)$$

satisfies the time independent Schrödinger equation and find the eigenvalue equation for $u(r)$.

b) For an infinite spherical well of radius $a$, find the ground state and first $\ell = 0$ excited state wave functions (up to a normalization constant) and the corresponding energies.

9. Consider a particle in a 1D box of length $a$ with $0 < x < a$ at the middle of the box. Calculate the first-order correction to the energies for all energy eigenstates due to the perturbation,

$$\hat{H}_{\text{int}} = \alpha \delta(x - a/2)$$

where $\delta()$ is the Dirac delta function and $\alpha$ is a constant.

10. For the $2P_{3/2}$ hydrogen multiplet (states $|n, \ell, j, m_j\rangle$ with $n = 2$, $L = 1$ and $J = 3/2$) calculate the correction due to the Zeeman effect in the weak-field approximation,

$$\hat{H}_B = \frac{eB}{2mc}(\hat{L}_z + 2\hat{S}_z).$$

Note: States of definite total angular momentum $|j, m_j\rangle$ are related to the product states of orbital angular momentum times spin $|\ell, m_{\ell}\rangle |s, m_s\rangle$ according to:

$$|\frac{3}{2}, \pm \frac{3}{2}\rangle = |1, \pm 1\rangle |\frac{1}{2}, \pm \frac{1}{2}\rangle$$

$$|\frac{3}{2}, \pm \frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|1, 1\rangle |\frac{1}{2}, -\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|1, 0\rangle |\frac{1}{2}, +\frac{1}{2}\rangle$$

$$|\frac{3}{2}, -\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|1, -1\rangle |\frac{1}{2}, +\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|1, 0\rangle |\frac{1}{2}, -\frac{1}{2}\rangle$$