Department of Physics and Astronomy, University of New Mexico

Quantum Mechanics Preliminary Examination

Fall 2017

Instructions:

- You should attempt all 10 problems (10 points each).
- Partial credit will be given if merited.
- NO cheat sheets are allowed.
- Total time: 3 hours.

Useful Constants, Formulas, and Relations:

- Neutron mass: $m_N = 1.67 \times 10^{-27}$ kg $= 940$ Mev/c^2
- Planck's constant: $h = 6.63 \times 10^{-34}$ Js, $\hbar c = 200$ eV.nm
- Bohr radius: $a_0 = 0.53 \times 10^{-10}$ m
- Ground state wavefunction of the Hydrogen atom:

$$\psi_{100} = \frac{1}{\pi^{1/2} a_0^{3/2}} e^{-r/2a_0}$$

- Harmonic oscillator lowering opertator:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p}_x \right)$$

- Ground state wavefunction of a harmonic oscillator with mass m and potential $V(x) = \frac{1}{2}kx^2$:

$$\psi_0(x) = \left(\frac{1}{\pi x_0^2} \right)^{1/4} \exp\left(-\frac{x^2}{2x_0^2} \right) \quad , \quad x_0 = \left(\frac{\hbar}{\sqrt{km}} \right)^{1/2}$$

- Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Rotation operator about direction $\hat{n}$ by an angle θ for a spin-1/2 particle:

$$\hat{R}(\phi\hat{n}) = \cos\left(\frac{\theta}{2}\right)\hat{I} - i\vec{\sigma}\cdot\hat{n}\,\sin\left(\frac{\theta}{2}\right)$$

- The radial Schrödinger equation for a central potential $V(r)$:

$$-\frac{\hbar^2}{2m}\frac{d^2\phi(r)}{dr^2} + \left(\frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right)\phi(r) = E\phi(r) \quad , \quad \phi(r) = r\psi(r)$$

- Gaussian integrals:

$$\int_{-\infty}^{+\infty} x^{2n}\exp(-ax^2) = \frac{(2n-1)!!}{2^n}\sqrt{\frac{\pi}{a^{2n+1}}}$$

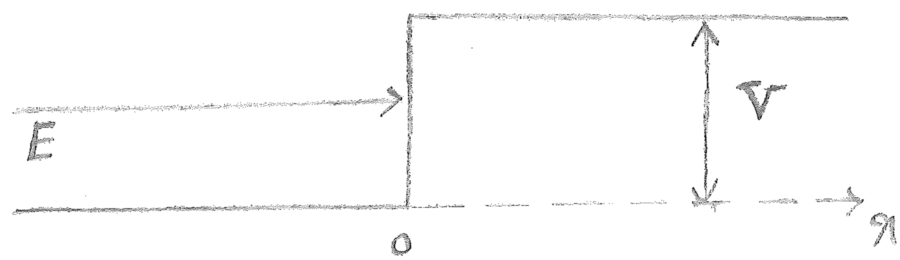
1– A polished silicon surface can act as an impenetrable barrier for neutrons. Suppose that a neutron is placed (negligible kinetic energy) above such a mirror with gravity acting down. Sketch the wave function overlaid with the potential. Estimate the height (in microns) that the neutron floats above the mirror.

2- Consider a two-state system with the following Hamiltonian:

$$\hat{H} = V\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right) + \Delta\left(-|1\rangle\langle 1| + |2\rangle\langle 2|\right),$$

where V and Δ are positive constants that have dimension of energy. The system is in the state $|1\rangle$ at $t = 0$. What is the probability that the system will be in state $|2\rangle$ at a later time t?

3– A free particle with mass m traveling to the right with energy E is reflected from a barrier of height V as shown in the figure. What is the ratio of probability for finding the particle in the interval $0 < x < l$ to that for finding it at $x > 0$?



4– Consider a simple harmonic oscillator in one dimension. Suppose that the oscillator is initially in a superposition of two energy eigenstates $|\psi\rangle = (|n+1\rangle + |n-1\rangle)/\sqrt{2}$. Determine the variance (i.e., undetainty squared) of the position as a function of time.

5- The nuclei of many large atoms are aspherical and their rotational motion is described by the Hamiltonian for an axially symmetric rotator:

$$\hat{H} = \frac{\hat{L}_x^2 + \hat{L}_y^2}{2I_1} + \frac{\hat{L}_z^2}{2I_2},$$

where $I_1 > I_2$. Find the ratio I_1/I_2 such that the energy levels of the $l = 2$ and $l = 3$ multiplets begin to overlap.

6- A harmonic oscillator with frequency ω is in its ground state. The mass of the oscillator is abruptly increased by a factor of 4. What is the probability that the new oscillator will be in its ground state?

7– An electron beam is prepared so that each particle is in the following spin state:

$$|\psi\rangle = \frac{\sqrt{3}}{2}|\uparrow\rangle - \frac{i}{2}|\downarrow\rangle\,,$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ respectively denote the spin-up and spin-down states in the z direction. Spin measurement along a new axis z' always yields $S_{z'} = \hbar/2$. What is the angle θ between the z' and z axes?

8- Two non-interacting identical spin-1/2 particles are confined within a one-dimensional box of length L. Assuming that both particles are in the n-th enery level, write down the complete wavefunction for the two-particle system.

9– Consider a particle of mass m that is in the free space outside a hard sphere with radius R. Derive the wavefunction of the particle for a state with energy E and zero angular momentum.

10– The hyperfine splitting of the ground state in the Hydrogen atom is due to the interaction of the electron spin and the proton spin and can be described by the following Hamiltonian:

$$\hat{H}_{\text{int}} = A\delta(\vec{r})(\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p)\,,$$

where $\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$ and A is a constant that is related to the mass and "g factor" of the electron and the proton. Obtain an expression for the hyperfine splitting in terms of A and Bohr radius a_0 to the lowest order. Using the fact that hyperfine transition results in a photon of wavelength $\lambda \approx 21$ cm, estimate the numerical value of A.