Preliminary Examination: Quantum Mechanics

Department of Physics and Astronomy
University of New Mexico

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Instructions:

- *the exam consists of 10 problems, 10 points each;*
- *partial credit will be given if merited;*
- *total time is 3 hours.*

Table of Constants and Conversion factors

Quantity	Symbol	$\approx$ value
speed of light	c	3.00×10^{8} m/s
Planck	h	6.63×10^{-34} J·s
electron charge	e	1.60×10^{-19} C
e mass	m_e	0.511 Mev/$c^2 = 9.11 \times 10^{-31}$ kg
p mass	m_p	938 Mev/$c^2 = 1.67 \times 10^{-27}$ kg
Bohr magneton	$e\hbar/(2m_e c)$	5.79×10^{-5}eV/T
fine structure	α	1/137
Boltzmann	k_B	1.38×10^{-23} J/K
Bohr radius	$a_0 = \hbar/(m_e c\alpha)$	0.53×10^{-10} m
AMU	u	931.5 Mev/$c^2 = 1.66 \times 10^{-27}$ kg
conversion constant	$\hbar c$	200 eV·nm = 200 MeV·fm
conversion constant	k_BT@ 300K	1/40 eV
conversion constant	1eV	1.60×10^{-19} J

Formulas

The Pauli spin matricies:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Rotation of spinor about $\hat{n}$ direction by an angle ϕ:

$$\hat{R}(\phi\hat{n}) = \cos\left(\frac{\phi}{2}\right)\hat{I} - i\vec{\sigma}\cdot\hat{n}\sin\left(\frac{\phi}{2}\right);$$

Angular momentum ($j = 0, \frac{1}{2}, 1, ...$):

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$$

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y; \quad [\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm$$

$$\hat{J}^2 |j,m\rangle = \hbar^2 j(j+1) |j,m\rangle$$

$$\hat{J}_z |j,m\rangle = \hbar m |j,m\rangle$$

$$\hat{J}_\pm |j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\, |j, m\pm 1\rangle$$

Hamiltonian for central potential for 2 particle system with reduced mass μ,

$$\hat{H} = \frac{\hat{P_r}^2}{2\mu} + \frac{\hat{L}^2}{2\mu r^2} + V(\hat{r}),$$

where the radial momentum operator is

$$\hat{P}_r = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r}r$$

$\ell = 1$ spherical harmonics:

$$Y_{1,\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\phi}\sin\theta = \mp\sqrt{\frac{3}{8\pi}}\frac{(x \pm iy)}{r}$$

$$Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$$

1. A polished silicon surface can act as an impenetrable barrier for neutrons. Suppose that a neutron is "placed" (negligible kinetic energy) above such a mirror with gravity acting down. Sketch the wave function overlaid with the potential. Estimate the height (in microns) that the neutron floats above the mirror. (For this estimate, use $m_N c^2 = 1000$ MeV, $\hbar c = 200$ eV-nm, and $m_N g = 10^{-13}\text{eV}/\mu\text{m}$.)

2. The carbon monoxide molecule absorbs radiation at a wavelength of 2.6 millimeters, corresponding to the excitation of the first rotational energy level from the ground state. The molecule can be taken to be a rigid rotor (dumbbell shape). Calculate the molecular bond length.

3. When we consider the two-electron states of Helium in perturbation theory, we take the e-e Coulomb interaction to be the perturbation.

   a) What is the unperturbed energy and degeneracy of the multiplet of first excited states including spin?
   b) Construct the states for $\ell = 0$, $s = 1$ and for $\ell = 1$, $s = 0$ as properly symmetrized linear combinations of $|space\rangle\,|spin\rangle$.
   c) Without doing any calculation, which (if any) of the states in (b) have the lower energy when including the perturbation? Give a physical argument justifying your answer.

4. Consider a system consisting of three orthonormal states

$$|1\rangle\,,\ |2\rangle\,,\ |3\rangle \text{ with } \langle j|i\rangle = \delta_{ij},$$

   and interacting via a time independent Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$ given in this basis by,

$$[H^0] = \begin{pmatrix} 2E_0 & 0 & 0 \\ 0 & E_0 & 0 \\ 0 & 0 & E_0 \end{pmatrix}$$

$$[H^1] = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 2\epsilon \\ 0 & 2\epsilon & \epsilon \end{pmatrix}$$

   where E_0 and ϵ are a real parameters with dimensions of energy. Calculate the energy eigenstates and eigenvalues in first order perturbation theory (first order in ϵ).

5. Consider two electrons produced in an entangled spin state with total spin $S = 0$ (spin singlet state). What is the probability to measure one electron with spin-up

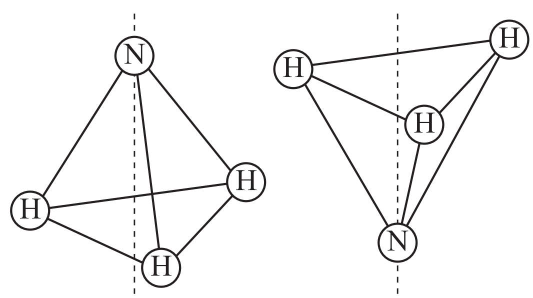
along the direction $\hat{a}$ and the other electron with spin-down along the direction $\hat{b}$, where these directions are arbitrary?

6. Consider the Ammonia molecule NH_3. The three hydrogens lie in a plane and form an isosceles triangle with the nitrogen along an axis perpendicular to the plane. The position-space wave function for the nitrogen moving in the potential of the three hydrogens has two linearly independent states: $|1\rangle$ and $|2\rangle$ corresponding to N above and below the plane of the hydrogens (see figure). The Hamiltonian in this basis has the form

$$[H] = \begin{bmatrix} E_0 & -A \\ -A & E_0 \end{bmatrix}$$

(a) Explain the off diagonal elements. What is their physical significance?

(b) If the system is in state $|1\rangle$ at $t = 0$, what is the probability to find the system in state $|2\rangle$ at time t?

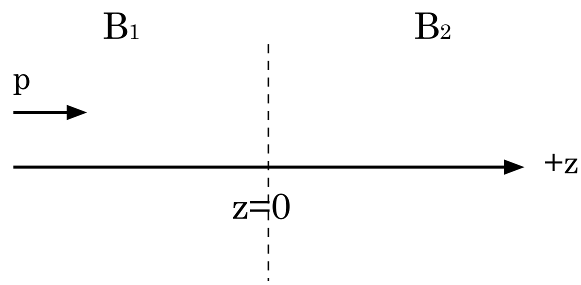


7. An electron with mass m moves in the z direction with spin-up ($S_z = +\frac{\hbar}{2}$) and parallel to a magnetic $\vec{B} = B\hat{z}$. In the region $z < 0$ the magnetic field is uniform with magnitude B_1, and for $z > 0$ it is again uniform, with magnitude B_2 (see figure). The time-independent Hamiltonian, is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{2\mu B_i}{\hbar}\hat{S}_z$$

where $i = 1, 2$ in the corresponding regions.

The Hamiltonian is effectively discontinuous at $z = 0$ assuming that the distance over which B changes is sufficiently short with respect to the electron's de Broglie wavelength. The electron is initially located to the left of the origin, and moves to the right, in the $+z$ direction, with momentum $\hbar k$. If its spin does not flip what is the probability that the electron will cross the origin and continue moving to the right? For what conditions will the probability be 0?



8. Consider a spin-1 particle of mass m, positive charge q and g-factor g in a magnetic field $\vec{B} = B\hat{z}$. Take the initial state at $t = 0$ to be an eigenstate of $\hat{S}_y$ with eigenvalue $+\hbar$ which is in terms of the z-basis eigenstates:

$$|\psi(0)\rangle = \frac{1}{2}\left(|1,1\rangle + i\sqrt{2}\,|1,0\rangle - |1,-1\rangle\right)$$

Consider only the spin part of the Hamiltonian,

$$\hat{H} = -\omega_0 \hat{S}_Z$$

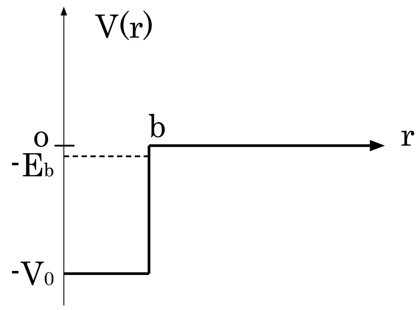
(a) What is the constant ω_0 in terms of the given constants? What is $|\psi(t)\rangle$?

(b) Calculate both $\langle\hat{S}_z\rangle$ and $\langle\hat{S}_x\rangle$ as functions of time.

The representation of $\hat{S}_x$ in the z-basis is

$$\hat{S}_x \rightarrow \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

9. A deuteron is bound state of proton and neutron ($m_p \approx m_n \approx m \approx 939$ MeV$/c^2$) with measured binding energy $E_b = 2.2$ MeV. There is only one bound state and it has zero angular momentum. The potential that binds the deuteron can be modeled as a three dimensional square well in the relative separation variable r, of depth V_0 and radius b (see Figure). The experimental value for the radius is $b = 1.7$ fm. In the limit that $E_b << V_0$ determine the value of V_0. ($\hbar c = 197$ MeV-fm)



10. Basic physics of neutrino oscillations can be illustrated by two-state flavor mixing. $|\nu_e\rangle, |\nu_\mu\rangle$ are flavor eigenstates and $|\nu_1\rangle, |\nu_2\rangle$ are mass eigenstates corresponding to neutrino masses m_1, m_2. Production and interaction of the neutrinos occurs via the flavor states, whereas the time evolution is determined by the mass eigenstates.

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$
$$|\nu_\mu\rangle = \sin\theta\,|\nu_1\rangle - \cos\theta\,|\nu_2\rangle$$

where θ parametrizes the magnitude of the mixing.

(a) Consider an electron neutrino produced with momentum $p >> m_i c$ where i refers to either mass. Find the energies E_1, E_2 for the neutrino states as functions of p by expanding the relativistic energy $(E^2 = m_i^2 c^4 + p^2 c^2)$ in a Taylor series.

(b) Since the neutrinos are ultra-relativistic, time is related to distance traveled L as $t = L/c$. Other than the energy expansion and letting $t = L/c$, the neutrino behaves as a simple non-relativistic two level quantum system. Show that the probability for the neutrino to change flavors from e to μ after propagating a distance L is

$$P = |\langle\nu_\mu|\nu_e(L)\rangle|^2 = \sin^2 2\theta \sin^2\left[\frac{(m_2^2 - m_1^2)c^2 L}{4p\hbar}\right]$$

(recall the trig. identity $2\sin\theta\cos\theta = \sin 2\theta$)