Preliminary Examination: Quantum Mechanics

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Instructions:

•The exam consists of 10 short-answer problems (10 points each).
•Where possible, show all work; partial credit will be given if merited.
•Personal notes on two sides of an 8×11 page are allowed.
•Total time: 3 hours.

Useful Formulae and Constants:

$$\int_{-\infty}^{\infty} dx \exp(-ax^2)x^{2n} = \frac{(2n-1)!!\sqrt{\pi}}{2^n a^{\frac{2n+1}{2}}}$$

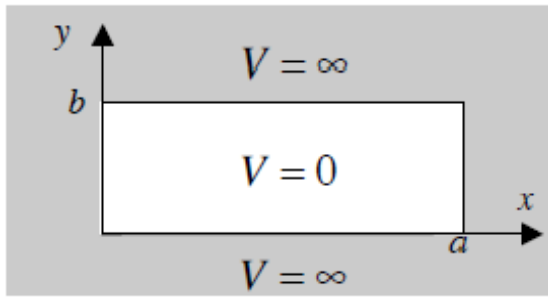
$$\psi_{0,0,0}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}} \exp\left(-\frac{r}{a_0}\right); a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$$

$$m = 9.1\times10^{-31}\text{kg},\ \hbar = 1.05\times10^{-34}\text{Js},\ \epsilon_0 = 8.8\times10^{-12}\ \frac{\text{C}^2}{\text{Nm}^2}, e = 1.6\times10^{-19}\text{C}$$

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z$$

$$\begin{aligned}\hat{x} &= \sqrt{\frac{\hbar}{2m\omega_0}}\left(b + b^\dagger\right)\\ \hat{p} &= -i\sqrt{\frac{\hbar m\omega_0}{2}}\left(b - b^\dagger\right)\end{aligned}$$

P1. A particle with mass m moving in two-dimensions is confined to an infinite rectangular-well potential having length a and width b. What are the energy eigenfunctions and associated eigenvalues of the system? Under what conditions are there degeneracies in the spectrum?



P2. A harmonic oscillator with frequency ω_0 is in the ground state. At time t_0 the frequency is abruptly reduced by a factor of two. For $t > t_0$, what is the probability that the new oscillator, with frequency $\omega_0/2$, will be in its ground state?

P3. Calculate the root-mean-square (RMS) speed of the electron in a hydrogen atom in the ground state (in m/s).

P4. An electron beam is prepared so that each particle is in the spin state $|\psi\rangle = \frac{1}{\sqrt{3}}|\uparrow\rangle + i\sqrt{\frac{2}{3}}|\downarrow\rangle$, where the states $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of $\hat{S}_z$. Along an axis z' a measurement of $\hat{S}_{z'}$ is made, and $S_{z'} = +\hbar/2$ is always found. ($\hat{S}_{z'} = -\hbar/2$ is never found.) What is the angle θ between the z' axis and the z axis?

P5. A particle's wave function is

$$\psi(x, y, z) = C(xy + 2yz)e^{-\alpha r}$$

where C and α are constants. What are the possible outcomes if L^2 is measured? What are the possible outcomes if L_z is measured? You may need the following spherical harmonics:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8}}\sin\theta e^{\pm i\varphi} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_2^{\pm 2} = \mp\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi} \quad Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\varphi} \quad Y_2^0 = \mp\sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

P6. A free electron is initially prepared in a Gaussian wave packet,

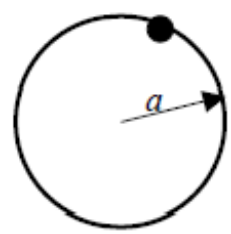
$$\psi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp(-\frac{x^2}{4\sigma^2})$$

with standard deviation $\sigma = 10$ μm in the x direction. Estimate the time (in seconds) for the uncertainty in the electron's position in x to increase to 20 μm.

P7. An electron with mass M is constrained to move on a ring of radius a in the xy plane. A uniform magnetic field B passes through the ring in the z direction. The Schroedinger equation is

$$\frac{1}{2M}\left(\frac{\hbar}{ia}\frac{\partial}{\partial\phi} - \frac{eBa}{2}\right)^2 \psi(\phi) = E\psi(\phi).$$

What are the energy eigenvalues E? Sketch a graph of the ground state energy as a function of B.



P8. A harmonic oscillator with frequency ω_0 and mass m is prepared in the state

$$|\psi\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

where α is a complex number, and $|n\rangle$ is an eigenstate of the number operator $\hat{n} = b^\dagger b$, where $b^\dagger$ and b are the raising and lowering operators. Show that $|\psi\rangle$ is an eigenstate of b, and find the associated eigenvalue. Find $\langle x(t)\rangle$.

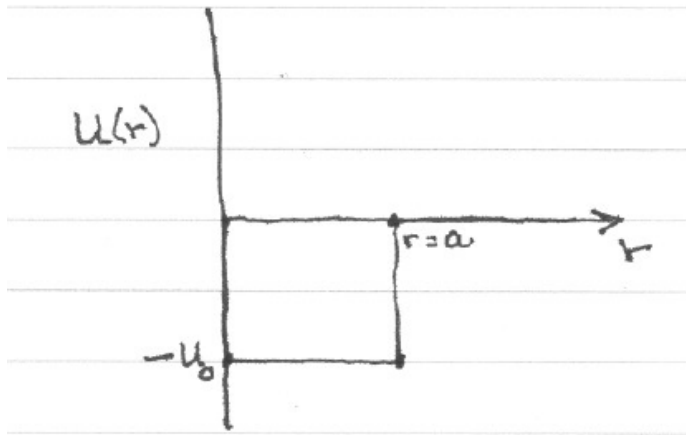
P9. Consider a particle with mass m in a three-dimensional spherical "square" well centered at the origin, described by the potential energy function,

$$U(r) = \begin{cases} -U_0; r < a \\ 0; r \geq a \end{cases},$$

sketched in the figure below. The radial Schroedinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}\Phi + \left(\frac{\hbar^2 \ell(\ell+1)}{2mr^2} + U(r)\right)\Phi = E\Phi$$

where ℓ is the angular momentum quantum number, and $\Phi(r) = r\Psi(r)$. What is the minimum depth U_0 so that there is at least one bound state?



P10. At $t = 0$, the wave function for a particle with mass m moving in the x direction through a channel with cross sectional area A is

$$\psi(x, y, z) \simeq \frac{1}{\sqrt{AL}} \left(\sin \frac{\pi x}{a} + i \sin \frac{2\pi x}{a} \right)$$

in the neighborhood of $x = 0$. Here L and a are constants. Use the probability flux associated with $\psi(x, y, z)$ to find the time rate of change in the probability to find the particle to the left of the origin $x = 0$, at $t = 0$.