Instructions:
• The exam consists of 10 short-answer problems (10 points each).
• Where possible, show all work; partial credit will be given if merited.
• Personal notes on two sides of an 8\times11 page are allowed.
• Total time: 3 hours.

Useful Formulae and Constants:
\[ \int_{-\infty}^{\infty} dx \exp(-ax^2)x^{2n} = \frac{(2n - 1)!!\sqrt{\pi}}{2^n a^{n+1/2}} \]
\[ \psi_{0,0,0}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}} \exp \left( -\frac{r}{a_0} \right); a_0 = \frac{4\pi \epsilon_0 \hbar^2}{me^2} \]

\[ m = 9.1 \times 10^{-31} \text{kg}, \quad \hbar = 1.05 \times 10^{-34} \text{Js}, \quad \epsilon_0 = 8.8 \times 10^{-12} \frac{C^2}{\text{Nm}^2}, \quad e = 1.6 \times 10^{-19} \text{C} \]

\[ \left[ \hat{S}_x, \hat{S}_y \right] = i\hbar \hat{S}_z \]

\[ \hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}} (b + b^\dagger) \]

\[ \hat{p} = -i\sqrt{\frac{\hbar m\omega_0}{2}} (b - b^\dagger) \]
P1. A particle with mass $m$ moving in two-dimensions is confined to an infinite rectangular-well potential having length $a$ and width $b$. What are the energy eigenfunctions and associated eigenvalues of the system? Under what conditions are there degeneracies in the spectrum?

P2. A harmonic oscillator with frequency $\omega_0$ is in the ground state. At time $t_0$ the frequency is abruptly reduced by a factor of two. For $t > t_0$, what is the probability that the new oscillator, with frequency $\omega_0/2$, will be in its ground state?

P3. Calculate the root-mean-square (RMS) speed of the electron in a hydrogen atom in the ground state (in m/s).

P4. An electron beam is prepared so that each particle is in the spin state $|\psi\rangle = \frac{1}{\sqrt{8}}|\uparrow\rangle + i\sqrt{\frac{3}{8}}|\downarrow\rangle$, where the states $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of $\hat{S}_z$. Along an axis $z'$ a measurement of $\hat{S}_{z'}$ is made, and $S_{z'} = +\hbar/2$ is always found. ($\hat{S}_{z'} = -\hbar/2$ is never found.) What is the angle $\theta$ between the $z'$ axis and the $z$ axis?

P5. A particle’s wave function is

$$\psi(x, y, z) = C(xy + 2yz)e^{-\alpha r}$$

where $C$ and $\alpha$ are constants. What are the possible outcomes if $L^2$ is measured? What are the possible outcomes if $L_z$ is measured? You may need the following spherical harmonics:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^1 = \frac{3}{\sqrt{8}} \sin \theta e^{i\phi}$$

$$Y_1^0 = \frac{3}{\sqrt{4\pi}} \cos \theta$$

$$Y_2^{\pm 1} = \frac{15}{\sqrt{32\pi}} \sin^2 \theta e^{\pm i\phi}$$

$$Y_2^0 = \frac{15}{\sqrt{8\pi}} \sin \theta \cos \theta e^{i\phi}$$

$$Y_2^\pm = \frac{5}{\sqrt{16\pi}} (3\cos^2 \theta - 1)$$
**P6.** A free electron is initially prepared in a Gaussian wave packet,

\[ \psi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{x^2}{4\sigma^2}\right) \]

with standard deviation \( \sigma = 10 \ \mu\text{m} \) in the \( x \) direction. Estimate the time (in seconds) for the uncertainty in the electron’s position in \( x \) to increase to \( 20 \ \mu\text{m} \).

**P7.** An electron with mass \( M \) is constrained to move on a ring of radius \( a \) in the \( xy \) plane. A uniform magnetic field \( B \) passes through the ring in the \( z \) direction. The Schrödinger equation is

\[ \frac{1}{2M} \left( \frac{\hbar}{ia} \frac{\partial}{\partial \phi} - \frac{eBa}{2} \right)^2 \psi(\phi) = E\psi(\phi). \]

What are the energy eigenvalues \( E \)? Sketch a graph of the ground state energy as a function of \( B \).

![Diagram of a ring with radius 'a' and magnetic field 'B' passing through it.]

**P8.** A harmonic oscillator with frequency \( \omega_0 \) and mass \( m \) is prepared in the state

\[ |\psi\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \]

where \( \alpha \) is a complex number, and \( |n\rangle \) is an eigenstate of the number operator \( \hat{n} = \hat{b}^\dagger \hat{b} \), where \( \hat{b}^\dagger \) and \( \hat{b} \) are the raising and lowering operators. Show that \( |\psi\rangle \) is an eigenstate of \( \hat{b} \), and find the associated eigenvalue. Find \( \langle x(t)\rangle \).

**P9.** Consider a particle with mass \( m \) in a three-dimensional spherical "square" well centered at the origin, described by the potential energy function,

\[ U(r) = \begin{cases} -U_0; & r < a \\ 0; & r \geq a \end{cases}, \]

sketched in the figure below. The radial Schrödinger equation is

\[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} \Phi \left( + \left( \frac{\hbar^2 \ell(\ell + 1)}{2mr^2} + U(r) \right) \Phi = E\Phi \right. \]

where \( \ell \) is the angular momentum quantum number, and \( \Phi(r) = r\Psi(r) \). What is the minimum depth \( U_0 \) so that there is at least one bound state?
P10. At $t = 0$, the wave function for a particle with mass $m$ moving in the $x$ direction through a channel with cross sectional area $A$ is

$$\psi(x, y, z) \simeq \frac{1}{\sqrt{AL}} \left( \sin \frac{\pi x}{a} + i \sin \frac{2\pi x}{a} \right)$$

in the neighborhood of $x = 0$. Here $L$ and $a$ are constants. Use the probability flux associated with $\psi(x, y, z)$ to find the time rate of change in the probability to find the particle to the left of the origin $x = 0$, at $t = 0$. 