# Preliminary Examination: Quantum Mechanics

*Department of Physics and Astronomy*

*University of New Mexico*

**Fall 2012**

**Instructions:**

- *the exam consists of 10 problems, 10 points each;*
- *partial credit will be given if merited;*
- *personal notes on two sides of* $8 \times 11$ *page are allowed;*
- *total time is 3 hours.*

Table of Constants and Conversion factors

| Quantity | Symbol | $\approx$ value |
|---|---|---|
| speed of light | c | $3.00 \times 10^{8}$ m/s |
| Planck | h | $6.63 \times 10^{-34}$ J·s |
| electron charge | e | $1.60 \times 10^{-19}$ C |
| e mass | $m_e$ | $0.511$ Mev$/c^2 = 9.11 \times 10^{-31}$ kg |
| p mass | $m_p$ | $938$ Mev$/c^2 = 1.67 \times 10^{-27}$ kg |
| fine structure | $\alpha$ | 1/137 |
| Boltzmann | $k_B$ | $1.38 \times 10^{-23}$ J/K |
| Bohr radius | $a_0 = \hbar/(m_e c\alpha)$ | $0.53 \times 10^{-10}$ m |
| Bohr magneton | $\mu_B$ | $5.8 \times 10^{-5}$eV/T |
| conversion constant | $\hbar c$ | 200 eV·nm |
| conversion constant | $k_B$T@ 300K | 1/40 eV |
| conversion constant | 1eV | $1.60 \times 10^{-19}$ J |

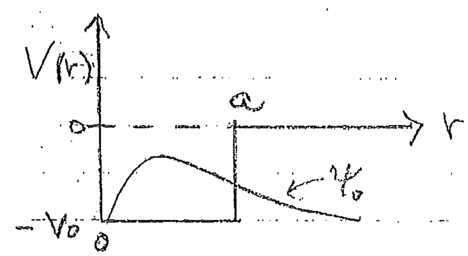
The Pauli spin matricies:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Angular momentum ($j = 0, \frac{1}{2}, 1, ....$):

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z$$

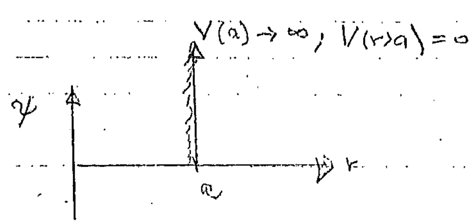
1. Consider a time independent but complex potential $V(x) = V_0(x) - i\Gamma$ where $\Gamma$ is independent of both space and time. Use the Schrödinger equation in 1D to show that the probability density is not conserved and find the time dependence of the wave function normalization (total probability).
2. For a 3D spherical well of depth $V_0$ and radius $a$.
(a) A bound state wave function is sketched on the figure below. On the same graph, sketch the ground state wave function in the limiting case wherein the particle is bound with infinitesimal energy.
(b) Find the minimum depth $V_0$ for the existence of a bound state. The sketch you made in (a) suggests a simple way to find the solution. If you must, you can always resort to solving the transcendental equation.




3. A particle with low momentum $\hbar k$ is scattered by a hard sphere of radius a.
(a) The radial wave function for the $\ell = 0$ partial wave is of the form $u(r) = rR(r) = \frac{1}{k}\sin(k\overset{r}{\cancel{a}} + \delta_0)$ where $\delta_0$ is the zeroth partial wave phase shift. Sketch the wave function on the figure below and determine $\delta_0$.
(b) What is the $k \to 0$ total cross section limit? Recall the total cross section is

$$\sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell$$

where $\delta_\ell$ is the $\ell^{th}$ partial wave phase shift.



4. Consider two electrons produced in an entangled spin state with total spin $S = 0$ (spin singlet state). What is the probability to measure one electron with spin-up along the direction $\hat{a}$ and the other electron with spin-up along the direction $\hat{b}$, where these directions are arbitrary?

5. Consider an electron bound to a metal with work function W. A constant electric field exists outside the metal, giving a potential $V(x) = W - e\mathcal{E}x$ for $x > 0$ where e is the absolute value of the electron charge and $\mathcal{E}$ is the magnitude of the electric field, and the direction $\hat{x}$ is taken normal to the metal surface. Find the tunneling transmission coefficient (T) for an electron at the Fermi energy (i.e. bound by energy W). You should get that $\log T \propto W^{3/2}/\mathcal{E}$. Recall that the transmission coefficient is of the form

$$T = \exp\left(\frac{-2}{\hbar}\int |p(x)|dx\right),$$

where $p(x)$ is the momentum.

6. A particle of mass m in a one-dimensional harmonic oscillator with angular frequency $\omega$ is prepared in a state where an energy measurement yields $\hbar\omega/2$ or $3\hbar\omega/2$ each with probability 1/2. If we also require that $\langle p \rangle = \sqrt{\hbar m\omega/2}$ at $t = 0$ the state is completely specified. Recall

$$\hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right).$$

(a) What is the complete normalized state as a function of time?
(b) What is $\langle p \rangle$ as a function of time?

7. The spinor (spin-$\frac{1}{2}$) rotation operator is given by

$$R(\alpha\hat{a}) = I\cos\left(\frac{\alpha}{2}\right) - i\hat{a}\cdot\vec{\sigma}\sin\left(\frac{\alpha}{2}\right)$$

about the direction $\hat{a}$ by angle $\alpha$, and where $\sigma_i$, $i = x, y, z$ are the Pauli spin matricies and I is the $2 \times 2$ unit matrix. Use this to construct the state $|+n\rangle$ corresponding to the particle having spin $+\frac{1}{2}$ along the direction given by the Euclidean vector $\hat{n} = \cos(\phi)\sin(\theta)\hat{x} + \sin(\phi)\sin(\theta)\hat{y} + \cos(\theta)\hat{z}$.

8. For a hydrogen atom in an electric field of strength $\mathcal{E}$ in the z direction has a perturbation $\hat{H}' = e\mathcal{E}z$. Show that there is no correction to the ground state to first order in perturbation theory.
9. A hydrogen atom in the presence of a magnetic field will have an additional interaction (Zeeman effect)

$$\hat{H}_{Zeeman} = \frac{\mu_B}{\hbar}\vec{B}\cdot\left(\hat{\vec{L}} + 2\hat{\vec{S}}\right)$$

where $\mu_B = 5.8 \times 10^{-5}$eV/T is the Bohr magneton.
(a) What condition on $B$ constitutes the weak field regime? What should $B$ be compared to?
(b) Find the corrections to the ${}^2P_{\frac{1}{2}}$ doublet ($j = 1, \ell = 1, s = 1/2,$) in the weak field approximation. Recall the explicit form of the states is:

$$|\frac{1}{2},\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|1,1\rangle|\frac{1}{2},-\frac{1}{2}\rangle - \sqrt{\frac{1}{3}}|1,0\rangle|\frac{1}{2},\frac{1}{2}\rangle$$

$$|\frac{1}{2},-\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|1,0\rangle|\frac{1}{2},-\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}|1,-1\rangle|\frac{1}{2},\frac{1}{2}\rangle$$

where the notation refers to $|j, m_j\rangle$ (the states of definite $j$) on the left hand side and $|1, m_\ell\rangle|\frac{1}{2}, m_s\rangle$ are the product of orbital and spin wave functions on the right hand side.
10. Use the variational method to estimate the ground state energy of the hydrogen atom ($V(r) = -e^2/r$). As a trial wave function take

$$\psi(\vec{r}) = \frac{A}{b^{3/2}}\left(1 - \frac{r}{b}\right), \text{ for } r < b,$$

$$\psi = 0 \text{ for } r > b$$

where the variational parameter b is a length and the dimensionless constant $A = \sqrt{30/4\pi}$. Notice that $\psi^*\psi d^3r$ is dimensionless as it should be. Express the variational parameter b in terms of the Bohr radius $a_0 = \hbar^2/me^2$ and the estimated energy in terms of $e^2/a_0 = 2(13.6eV)$ Recall the Laplacian in spherical coordinates is:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$