Department of Physics and Astronomy, University of New Mexico

E&M Preliminary Examination

Spring 2013

Instructions:

- The exam consists of 10 problems (10 pts each).
- Partial credit will be given if merited.
- Personal notes on the two sides of an 8.5"x 11" sheet are allowed.
- Total time: 3 hours.

Possibly Useful Formulas

- Relation of spherical polar coordinates, (r, θ, ϕ), to Cartesian coordinates:

$$x = r\,\sin\theta\,\cos\phi, \quad y = r\,\sin\theta\,\sin\phi, \quad z = r\,\cos\theta.$$

  Unit vectors:

$$\hat{r} = \sin\theta\,\cos\phi\,\hat{x} + \sin\theta\,\sin\phi\,\hat{y} + \cos\theta\,\hat{z};$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}; \quad \hat{\theta} = \hat{\phi} \times \hat{r}.$$

- Laplacian in spherical polar coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

- Electric field at position $\vec{r}$ due to a point electric dipole of moment $p\hat{z}$ located at the origin:

$$\vec{E}(\vec{r}) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right),$$

  where $\hat{r}$ and $\hat{\theta}$ are two unit vectors of the spherical polar coordinate system.

- Stokes' theorem:

$$\oint_C \vec{A}\cdot d\vec{l} = \iint_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,da,$$

where S is an open surface bounded by the closed curve C.

- Biot-Savart Law for the magnetic field at position $\vec{r}$ due to a steady current element $Id\vec{\ell}'$ located at position $\vec{r}'$:

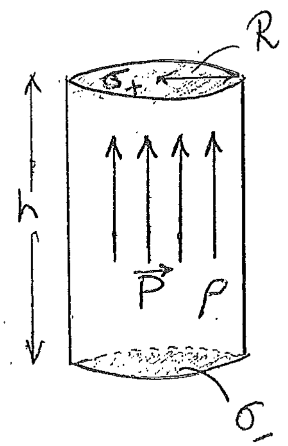
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{Id\vec{\ell}'\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}.$$

- Fresnel formulas for the amplitude reflection coefficient of a plane wave incident at a planar interface between two dielectrics:

$$r_\perp = \frac{n\cos\theta - n'\cos\theta'}{n\cos\theta + n'\cos\theta'};\quad r_\parallel = \frac{n'\cos\theta - n\cos\theta'}{n'\cos\theta + n\cos\theta'},$$

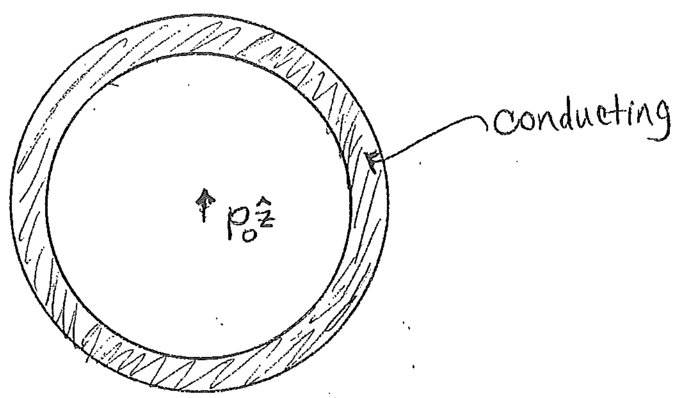
where $\perp$, $\parallel$ refer, respectively, to polarizations perpendicular and parallel to the plane of incidence. The angles of incidence and refraction are θ and θ', and n, n' are the refractive indices of the medium of incidence and the medium of transmission, respectively.

1. A solid cylinder of radius R and height h is polarized along its axis with polarization density $\vec{P}$ increasing linearly with height from a value $P_1\hat{z}$ at the bottom face to $P_2\hat{z}$ at the top face. What are the bound surface and volume charge densities, labeled as σ_+, σ_-, and ρ in the figure below? Show by integrating the total bound charge densities that there is no net charge in the cylinder.



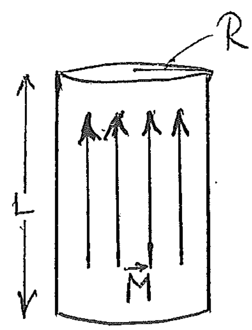
2. From the uniform bound charge densities in the previous problem, calculate the electric field at the center of the cylinder.

3. A perfectly conducting spherical shell of inner radius R encloses a point electric dipole of moment $\vec{p}_0 = p_0\hat{z}$ at its center. The electric field at any point in the empty space enclosed by the shell, as one may show, is equal to the vector sum of the field at that point due to the point dipole and a uniform electric field $E_0\hat{z}$ parallel to the dipole orientation. (The latter is the contribution to the total field from charges induced on the shell surface.) Calculate the value of E_0 such that the net electric field in the interior is normally oriented at any point on the inner surface of the shell, as required by the conducting boundary condition. (*Hint:* The Cartesian unit vector $\hat{z}$ may be expressed as $\cos\theta\hat{r} - \sin\theta\hat{\theta}$ in terms of spherical basis vectors.)



4. The electrostatic potential of an electric dipole of moment $\vec{p}$ at a point a vector separation $\vec{r}$ away has the form $V = \vec{p}\cdot\vec{r}/(4\pi\epsilon_0 r^3)$. Show by calculating its Laplacian that this potential obeys the Laplace equation at all points except where the dipole is located.

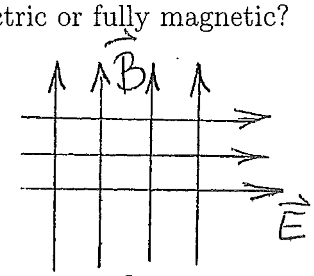
5. Consider the magnetic field inside a cylindrical bar magnet of radius R and height L. Let its magnetization $\vec{M}$ be uniform. Calculate the magnetic field at the center of the magnet in two extreme limits, $L >> R$ and $L << R$.



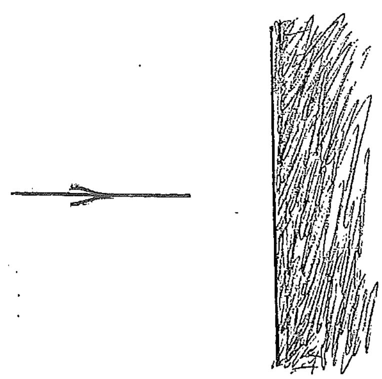
6. A charge q moving relativistically through interstellar space finds itself caught in a magnetic field $\vec{B}$. Describe the possible forms of motion of the charge, assuming the magnetic field to be spatially uniform. By choosing a suitable origin of coordinates, show that the components of the linear momentum of the charge at position $\vec{r}$ that are transverse to $\vec{B}$ may be expressed as $\vec{p}_{\perp} = q\vec{r} \times \vec{B}$. Identify explicitly your choice of the origin.

7. For the previous problem, show by means of the Stokes theorem that the line integral of the linear momentum vector of the charge along *any* closed path C, *i.e.*, $\oint_C \vec{p} \cdot d\vec{l}$, is proportional to the magnetic flux threading the path. Calculate the constant of proportionality.

8. Show that under the combined action of a constant electric field and a constant magnetic field, a charge q when moving with a certain velocity experiences no net force. For $\vec{E} = E\hat{x}$ and $\vec{B} = B\hat{y}$, with $E < cB$, calculate the possible velocities of the charge for which it experiences no net force. Can this happen when $E > cB$? If not, then describe the nature of the motion of the charge in this case. For each of the two cases, is there a Lorentz frame in which the complete electromagnetic field is either fully electric or fully magnetic?



9. A monochromatic plane electromagnetic wave of angular frequency ω is incident normally from vacuum onto the surface of a material with a complex dielectric permittivity $\epsilon' + i\epsilon''$, where $\epsilon' >> \epsilon''$ are both real and positive. For a thick material, calculate the fraction of radiation energy that is (i) reflected, (ii) transmitted, and (iii) absorbed by the material. Calculate also the characteristic penetration depth of the radiation into the material.



10. Consider a classical model of a hydrogen atom in which an electron (mass: m, charge: $-e$) orbits around an infinitely massive proton in a circular orbit of radius r in a plane. That such an atom must be unstable follows from the fact that the electron, since it is accelerated in circular motion, must radiate energy continuously according to the Larmor radiation formula,

$$P = \frac{e^2 a^2}{6\pi\epsilon_0 c^3},$$

where a is its acceleration. Neglect radiation first and use Newton's second law to determine a in terms of e, m, and r. Assume non-relativistic motion. What is the relation between the total mechanical energy U (beware it is negative!) of the electron and its orbital radius r? Now apply the energy conservation law,

$$dU/dt = -P,$$

to this relation to derive a simple first-order differential equation connecting r to the time t. Solve that equation to show that in an atom of initial orbital radius a_0 the electron will fall into the proton in a time equal to $a_0^3/(4r_0^2 c)$, where $r_0 \equiv e^2/(4\pi\epsilon_0 mc^2)$ is the so-called classical radius (not orbital radius) of the electron. What assumption(s) about the classical atom, as set out in this problem, might eventually break down even at the classical level as the electron spirals into the proton?